

SHARP ADAMS–MOSER–TRUDINGER TYPE INEQUALITIES IN THE HYPERBOLIC SPACE

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ABSTRACT. The purpose of this paper is to establish some Adams–Moser–Trudinger inequalities, which are the borderline cases of the Sobolev embedding, in the hyperbolic space \mathbb{H}^n . First, we prove a sharp Adams inequality of order two with the exact growth condition in \mathbb{H}^n . Then we use it to derive a sharp Adams-type inequality and an Adachi–Tanaka-type inequality. We also prove a sharp Adams-type inequality with Navier boundary condition on any bounded domain of \mathbb{H}^n , which generalizes the result of Tarsi to the setting of hyperbolic spaces. Finally, we establish a Lions-type lemma and an improved Adams-type inequality in the spirit of Lions in \mathbb{H}^n . Our proofs rely on the symmetrization method extended to hyperbolic spaces.

1. INTRODUCTION

In the literature, Sobolev spaces, geometric and analytic inequalities can be considered as one of the central tools in many areas such as analysis, differential geometry, partial differential equations, calculus of variations, etc. Of importance, among these inequalities, are the classical Sobolev inequalities which assert that the following embedding $W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for $n \geq 2$, $kp < n$, and $1 \leq q \leq np/(n - kp)$ where Ω is a bounded domain in \mathbf{R}^n . However, in the limit case $kp = n$, we can easily show by many examples that $W_0^{k,n/k}(\Omega) \not\subset L^\infty(\Omega)$. In this special situation, the so-called Moser–Trudinger and Adams inequalities are the perfect replacements; see [Tru67, Mos70, Ada88].

It is now widely recognized that the Moser–Trudinger and Adams inequalities have played so many important roles and has been widely used in geometric analysis and PDE; for example, we refer the reader to [CT03, LL12a, LL12b, LL14, Sha87, TZ00] and references therein.

This remarkable inequality has also been generalized in many directions. For instance, the singular Moser–Trudinger inequality was discovered in [AS07], the best constant for Moser–Trudinger inequality on domains of finite measure on the Heisenberg group was found in [CL01, LLT12]. There has also been substantial progress for the Moser–Trudinger inequality on Euclidean spheres, CR spheres, as well as on compact Riemannian manifolds, hyperbolic spaces; see [Bec93, CL01, CL04, Fon93, Li05, LT13].

For the question of the existence of optimizer functions for the Moser–Trudinger inequality, it was first addressed by Carleson and Chang [CC86] on balls in the Euclidean space. Then this result was extended to arbitrary smooth domains by Flucher [Flu92] and Lin [Lin96].

1.1. Moser–Trudinger and Adams inequalities on \mathbf{R}^n .

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1.1.1. *Moser–Trudinger inequalities on \mathbf{R}^n .* Speaking of the Moser–Trudinger inequality on bounded domains, it was established independently by Yudovič [Yud61], Pohožaev [P65], and Trudinger [Tru67]. Later, by sharpening the Trudinger inequality, Moser proved that there exists a sharp constant

$$\alpha_n = n^{n/(n-1)} \Omega_{n-1}^{1/(n-1)},$$

where Ω_n denotes the volume of the unit ball \mathbb{B}^n in \mathbf{R}^n , such that

$$\sup_{\substack{u \in C_0^\infty(\Omega): \\ \int_\Omega |\nabla u|^n dx \leq 1}} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha |u|^{n/(n-1)}) dx \leq c_0 < +\infty \quad (\text{MT}_b^{\mathbf{R}})$$

for any $\alpha \leq \alpha_n$ and for any bounded domain Ω in \mathbf{R}^n . In $(\text{MT}_b^{\mathbf{R}})$, the constant α_n is sharp in the sense that if $\alpha > \alpha_n$, then the supremum above will become infinity.

When Ω has infinite measure, the sharp version of Moser–Trudinger-type inequality for unbounded domain, or subcritical Moser–Trudinger inequality, was established by Adachi and Tanaka [AT99]. To be more precise, they proved that for any $\alpha \in (0, \alpha_n)$ there exists a constant $C(n, \alpha) > 0$ such that

$$\sup_{\substack{u \in W^{1,n}(\mathbf{R}^n) \setminus \{0\}: \\ \int_{\mathbf{R}^n} |\nabla u|^n dx \leq 1}} \frac{1}{\|u\|_{L^n(\mathbf{R}^n)}^n} \int_{\mathbf{R}^n} \Phi_{n,1}(\alpha |u|^{n/(n-1)}) dx \leq C(n, \alpha), \quad (\text{MT}_{us}^{\mathbf{R}})$$

where

$$\Phi_{n,1}(t) = e^t - \sum_{j=0}^{n-2} t^j / j!.$$

The constant α_n , appearing in the Moser–Trudinger inequality $(\text{MT}_b^{\mathbf{R}})$, is also sharp in the sense that if $\alpha \geq \alpha_n$, then the supremum above is infinite. The question is: *What happens when $\alpha = \alpha_n$?*

When $\alpha = \alpha_n$, the critical Moser–Trudinger inequality for any unbounded domain in \mathbf{R}^n was proved by Ruf [Ruf05] for $n = 2$ and by Li and Ruf [LR08] for the case $n \geq 2$. This inequality asserts that there is a constant $C(n) > 0$ such that for any domain $\Omega \subset \mathbf{R}^n$, there holds

$$\sup_{\substack{u \in W_0^{1,n}(\Omega): \\ \|u\|_{W_0^{1,n}(\Omega)} \leq 1}} \int_\Omega \Phi_{n,1}(\alpha_n |u|^{n/(n-1)}) dx \leq C(n), \quad (\text{MT}_{uc}^{\mathbf{R}})$$

where

$$\|u\|_{W_0^{1,n}(\Omega)} = (\|\nabla u\|_{L^n(\mathbf{R}^n)}^n + \|u\|_{L^n(\mathbf{R}^n)}^n)^{1/n}.$$

In addition, it was found that the same constant α_n is sharp in the sense that the supremum above will be infinite if α_n is replaced by any $\alpha > \alpha_n$.

The existence of optimizer functions for the Moser–Trudinger inequality in entire space was studied in [Ish11, LR08, Ruf05]. More recently, sharp Moser–Trudinger inequalities has been established on the entire Heisenberg group at the critical case in [LL12c], at the subcritical case in [LLT14], or in weighted form in Heisenberg-type group in [LT13] where symmetrization argument is not available.

We note that there is a fundamental difference between $(\text{MT}_{us}^{\mathbf{R}})$ and $(\text{MT}_{uc}^{\mathbf{R}})$. In fact, inequality $(\text{MT}_{us}^{\mathbf{R}})$ only holds for $\alpha < \alpha_n$ while inequality $(\text{MT}_{uc}^{\mathbf{R}})$ holds for all $\alpha \leq \alpha_n$. The reason behind this difference is that in $(\text{MT}_{us}^{\mathbf{R}})$ we require functions with the L^n -norm of their gradient less than or equal to 1 while in $(\text{MT}_{uc}^{\mathbf{R}})$, we require functions with $W^{1,n}$ -norm less than or equal to 1. In other word, the failure of the original Moser–Trudinger inequality $(\text{MT}_b^{\mathbf{R}})$ on the entire space \mathbf{R}^n can be recovered either by weakening the exponent α_n as in $(\text{MT}_{us}^{\mathbf{R}})$ or by strengthening the Dirichlet norm $\|\nabla u\|_{L^n(\mathbf{R}^n)}$ as in $(\text{MT}_{uc}^{\mathbf{R}})$.

A natural question arises: *Can we still achieve the best constant α_n when we only require the condition $\|\nabla u\|_{L^n(\mathbf{R}^n)} \leq 1$?* This question was answered by Ibrahim, Masmoudi and Nakanishi [IMN15] for the case $n = 2$ and by Masmoudi and Sani [MS15] for arbitrary $n \geq 2$. In their works, they proved the following inequality

$$\sup_{\substack{u \in W^{1,n}(\mathbf{R}^n) \setminus \{0\}: \\ \|\nabla u\|_{L^n(\mathbf{R}^n)} \leq 1}} \frac{1}{\|u\|_{L^n(\mathbf{R}^n)}^n} \int_{\mathbf{R}^n} \frac{\Phi_{n,1}(\alpha_n |u|^{n/(n-1)})}{(1 + |u|)^{n/(n-1)}} dx < \infty. \quad (\text{MT}_{uh}^{\mathbf{R}})$$

Moreover, this inequality is sharp in the sense that it fails if the power $n/(n-1)$ in the denominator is replaced by any $p < n/(n-2)$.

1.1.2. *Adams inequalities on \mathbf{R}^n .* In the seminal work [Ada88], Adams extended the Moser–Trudinger inequality ($\text{MT}_b^{\mathbf{R}}$) to higher order Sobolev spaces $W_0^{m,n/m}(\Omega)$ where $\Omega \subset \mathbf{R}^n$ is of finite measure. Let m be a positive integer less than n , we denote the m th order gradient of a function u on \mathbf{R}^n by

$$\nabla^m u = \begin{cases} \Delta^{m/2} u & \text{if } m \text{ is even,} \\ \nabla \Delta^{(m-1)/2} u & \text{if } m \text{ is odd.} \end{cases}$$

Then Adams [Ada88] proved that there exists a constant $C(n, m) > 0$ such that the following inequality holds

$$\sup_{\substack{u \in W_0^{m,n/m}(\Omega): \\ \int_{\Omega} |\nabla^m u|^{n/m} dx \leq 1}} \frac{1}{|\Omega|} \int_{\Omega} \exp(\beta(n, m) |u|^{n/(n-m)}) dx \leq C(n, m), \quad (\text{A}_b^{\mathbf{R}})$$

where the constant $\beta(n, m)$ is given as follows

$$\beta(n, m) = \begin{cases} \Omega_n^{-1} \left(\pi^{n/2} 2^m \frac{\Gamma((m+1)/2)}{\Gamma((n-m+1)/2)} \right)^{n/(n-m)} & \text{if } m \text{ is odd,} \\ \Omega_n^{-1} \left(\pi^{n/2} 2^m \frac{\Gamma(m/2)}{\Gamma((n-m)/2)} \right)^{n/(n-m)} & \text{if } m \text{ is even.} \end{cases}$$

Moreover, the constant $\beta(n, m)$ in ($\text{A}_b^{\mathbf{R}}$) is sharp in the sense that if we replace it by any $\beta > \beta(n, m)$, then the supremum above will become infinite.

Adams inequality ($\text{A}_b^{\mathbf{R}}$) on domains of finite measure was recently extended by Tarsi [Tar12] to a larger space, namely the Sobolev space with homogeneous Navier boundary condition $W_N^{m,n/m}(\Omega)$,

$$W_N^{m,n/m}(\Omega) = \{u \in W^{m,n/m} : \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } 0 \leq j \leq [(m-1)/2]\}.$$

Note that $W_N^{m,n/m}(\Omega)$ contains $W_0^{m,n/m}(\Omega)$ as proper, closed subspace.

Sharp Adams inequality on entire space \mathbf{R}^n was first proved by Ruf and Sani [RS13] for the case that m is even and by Lam and Lu [LL12d] for the remaining case when m is odd. Their results read as follows: Let m be an integer less than n , for each $u \in W_0^{m,n/m}(\Omega)$ we denote

$$\|u\|_{m,n} = \begin{cases} \|(-\Delta + I)^{m/2} u\|_{L^{n/m}(\Omega)} & \text{if } m \text{ is even,} \\ \left(\|\nabla(-\Delta + I)^{(m-1)/2} u\|_{L^{n/m}(\Omega)}^{n/m} + \|(-\Delta + I)^{(m-1)/2} u\|_{L^{n/m}(\Omega)}^{n/m} \right)^{m/n} & \text{if } m \text{ is odd,} \end{cases}$$

then there holds

$$\sup_{\substack{u \in W_0^{m,n/m}(\Omega): \\ \|u\|_{m,n} \leq 1}} \int_{\Omega} \Phi_{n,m}(\beta(n, m) |u|^{n/(n-m)}) dx < +\infty, \quad (\text{A}_{bc}^{\mathbf{R}})$$

where

$$\Phi_{n,m}(t) = e^t - \sum_{j=0}^{j_{n/m}-2} t^j / j!$$

with

$$j_{n/m} = \min \{j \in \mathbb{N}, j \geq n/m\}.$$

In addition, the constant $\beta(n, m)$ in (A_{bc}^R) is sharp in the sense that if we replace $\beta(n, m)$ in (A_{bc}^R) by any $\beta > \beta(n, m)$, then the supremum in (A_{bc}^R) will be infinite. We refer the reader to [LL13] for a sharp Adams-type inequality of fractional order $\alpha \in (0, n)$ where a rearrangement-free argument was used.

Recently, Masmoudi and Sani [MS14] proved a sharp Adams inequality with exact growth condition in \mathbb{R}^4 . Then Lu, Tang and Zhu [LTZ15] extended the result of Masmoudi and Sani to all dimension $n \geq 2$ to assert the existence of the constant $C(n) > 0$ such that

$$\sup_{\substack{u \in W^{2,n/2}(\mathbb{R}^n) \setminus \{0\}: \\ \|\nabla^2 u\|_{L^{n/2}(\mathbb{R}^n)} \leq 1}} \frac{1}{\|u\|_{L^{n/2}(\mathbb{R}^n)}^{n/2}} \int_{\mathbb{R}^n} \frac{\Phi_{n,2}(\beta(n, 2)|u|^{n/(n-2)})}{(1 + |u|)^{n/(n-2)}} dx < +\infty. \quad (A_e^R)$$

Moreover, the power $n/(n-2)$ in the denominator is sharp in the sense that the supremum above will become infinite if we replace the power in the denominator by any $p < n/(n-2)$. In applications, inequality (A_e^R) implies a subcritical sharp Adams inequality in the spirit of Adachi and Tanaka which strengthens an inequality of Ogawa and Ozawa [OO91]. It also implies a sharp Adams-type inequality under the norm

$$\|u\|_{W^{2,n/2}} = (\|u\|_{L^{n/2}(\mathbb{R}^n)}^{n/2} + \|\Delta u\|_{L^{n/2}(\mathbb{R}^n)}^{n/2})^{2/n},$$

namely

$$\sup_{\substack{u \in W^{2,n/2}(\mathbb{R}^n): \\ \|u\|_{W^{2,n/2}} \leq 1}} \int_{\mathbb{R}^n} \Phi_{n,2}(\beta(n, 2)|u|^{n/(n-2)}) dx < +\infty. \quad (1.1)$$

The constant $\beta(n, 2)$ is sharp; see [LTZ15, MS14] for more details. A version of higher order derivatives of (1.1) has recently been proved by Fontana and Morpurgo in [FM15]. We remark that a version of higher order derivatives of (MT_{uh}^R) and (A_e^R) is still unknown; however, a weaker result can be found in [FM15].

1.2. Moser–Trudinger and Adams inequalities on \mathbb{H}^n . Although there have been extensive works on the best constants for Moser–Trudinger and Adams inequalities in Euclidean spaces, Heisenberg groups, and compact Riemannian manifolds as listed above, much less is known for the sharp constants for Moser–Trudinger and Adams inequalities on hyperbolic spaces.

The hyperbolic space \mathbb{H}^n with $n \geq 2$ is a complete, simply connected Riemannian manifold having constant sectional curvature equal to -1 , and for a given dimensional number, any two such spaces are isometries [Wol67]. There is a number of models for \mathbb{H}^n , however, the most important models are the half-space model, the ball model, and the hyperboloid or Lorentz model. In this paper, we will use the ball model since this model is especially useful for questions involving rotational symmetry. Given $n \geq 2$, we denote by B_n the open unit ball in \mathbb{R}^n . Clearly, B_n can be endowed with the Riemannian metric

$$g(x) = \sum_{i=1}^n \left(\frac{2}{1 - |x|^2} \right)^2 dx_i^2,$$

which is called the ball model of the hyperbolic space \mathbb{H}^n . The volume element of \mathbb{H}^n is given by

$$dV_g(x) = \left(\frac{2}{1 - |x|^2} \right)^n dx,$$

where dx denotes the Lebesgue measure in \mathbb{R}^n . For any subset $E \subset B_n$, we denote $|E| = \int_E dV_g$. Let $d(0, x)$ denote the hyperbolic distance between the origin and x . It is well-known that $d(0, x) = \ln((1 + |x|)/(1 - |x|))$ for arbitrary $x \in B_n$. In this new context, we still

use ∇ and Δ to denote the Euclidean gradient and Laplacian as well as $\langle \cdot, \cdot \rangle$ to denote the standard inner product in \mathbf{R}^n . Then, in terms of ∇ , Δ , and $\langle \cdot, \cdot \rangle$, the hyperbolic gradient ∇_g and the Laplace–Beltrami operator Δ_g are given by

$$\nabla_g = \left(\frac{1 - |x|^2}{2} \right)^2 \nabla, \quad \Delta_g = \left(\frac{1 - |x|^2}{2} \right)^2 \Delta + (n - 2) \frac{1 - |x|^2}{2} \langle x, \nabla \rangle.$$

Given a bounded domain $\Omega \subset \mathbb{H}^n$, we denote

$$\|f\|_{p,\Omega} = \left(\int_{\Omega} |f|^p dV_g \right)^{1/p}$$

for each $1 \leq p < \infty$. Then we have the following

$$\|\nabla_g f\|_{n,\Omega} = \left(\int_{\Omega} \langle \nabla_g f, \nabla_g f \rangle_g^{n/2} dV_g \right)^{1/n} = \left(\int_{\Omega} |\nabla f|^n dx \right)^{1/n}.$$

In the case $\Omega = \mathbb{H}^n$, we simply write $\|f\|_p$ instead of $\|f\|_{p,\mathbb{H}^n}$ for all $1 \leq p < \infty$. Throughout the paper, we also use $W_0^{2,n/2}(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W_0^{2,n/2}(\Omega)} = \left(\int_{\Omega} |u|^{n/2} dV_g + \int_{\Omega} |\Delta_g u|^{n/2} dV_g \right)^{2/n}.$$

In particular, we will denote $W^{2,n/2}(\mathbb{H}^n)$ as the completion of $C_0^\infty(\mathbb{H}^n)$ under the norm

$$\|u\|_{W^{2,n/2}(\mathbb{H}^n)} = \left(\int_{\mathbb{H}^n} |u|^{n/2} dV_g + \int_{\mathbb{H}^n} |\Delta_g u|^{n/2} dV_g \right)^{2/n}.$$

1.2.1. Moser–Trudinger inequalities on \mathbb{H}^n . In [MS10], Mancini and Sandeep established a sharp Moser–Trudinger inequality on the 2-dimensional hyperbolic space \mathbb{B}_2 . They proved

$$\sup_{\substack{u \in C_0^\infty(B_2): \\ \|\nabla_g u\|_2 \leq 1}} \int_{B_2} (e^{4\pi u^2} - 1) dV_g < \infty, \quad (\text{MT}_b^{\mathbb{H}^2})$$

where the constant $4\pi^2$ is sharp in the sense that the supremum above will be infinite if $4\pi^2$ is replaced by any number larger than $4\pi^2$. The Moser–Trudinger inequality on bounded domains Ω in any hyperbolic space of any higher dimension was proved by Lu and Tang [LT13]

$$\sup_{\substack{u \in C_0^\infty(\Omega): \\ \|\nabla_g u\|_{n,\Omega} \leq 1}} \frac{1}{|\Omega|} \int_{\Omega} \exp(\alpha_n |u|^{n/(n-1)}) dx < \infty \quad (\text{MT}_b^{\mathbb{H}})$$

with the sharp constant α_n . We note that the best constant in the Moser–Trudinger inequality on bounded domain in hyperbolic space ($\text{MT}_b^{\mathbb{H}}$) is similar to the one of the Moser–Trudinger inequality on bounded domain in Euclidean space ($\text{MT}_b^{\mathbb{R}}$).

When Ω has infinite volume, a sharp subcritical Moser–Trudinger-type inequality in spirit of Adachi–Tanaka was recently proved by Lu and Tang in [LT13]. They show that for any $\alpha \in (0, \alpha_n)$, there exists a constant $C_\alpha > 0$ such that

$$\sup_{\substack{u \in W^{1,n}(\mathbb{H}^n) \setminus \{0\}: \\ \|\nabla_g u\|_n \leq 1}} \frac{1}{\|u\|_n^n} \int_{\mathbb{H}^n} \Phi_{n,1}(\alpha |u|^{n/(n-1)}) dV_g \leq C_\alpha, \quad (\text{MT}_{us}^{\mathbb{H}})$$

and the constant α_n is sharp in the sense that for $\alpha \geq \alpha_n$, the supremum in ($\text{MT}_{us}^{\mathbb{H}}$) will be infinite.

It was also established in [LT13] a sharp critical Moser–Trudinger inequality on the entire hyperbolic space when we restrict the norms of functions to full hyperbolic Sobolev

norm, namely any $\tau > 0$, there exists a constant $C_{n,\tau} > 0$ such that

$$\sup_{\substack{u \in W^{1,n}(\mathbb{H}^n): \\ \|\nabla_g u\|_n^n + \tau \|u\|_n^n \leq 1}} \int_{\mathbb{H}^n} \Phi_n(\alpha_n |u|^{n/(n-1)}) dV_g \leq C_{n,\tau}. \quad (\text{MT}_{uc}^{\mathbb{H}})$$

The constant α_n is sharp in the sense that the supremum above will become infinite if α_n is replaced by any $\alpha > \alpha_n$. In view of $(\text{MT}_{us}^{\mathbb{H}})$ and $(\text{MT}_{uc}^{\mathbb{H}})$, a natural question, as in the Euclidean space, arises: *Can we still achieve the best constant α_n when we only require the restriction on the norm $\|\nabla_g u\|_n \leq 1$?* This question also was answered in [LT15] by Lu and Tang. They proved a sharp Moser–Trudinger inequality with exact growth condition in hyperbolic space as follows

$$\sup_{\substack{u \in W^{1,n}(\mathbb{H}^n) \setminus \{0\}: \\ \|\nabla_g u\|_n \leq 1}} \frac{1}{\|u\|_n^n} \int_{\mathbb{H}^n} \frac{\Phi_{n,1}(\alpha_n |u|^{n/(n-1)})}{(1 + |u|)^{n/(n-1)}} dV_g < +\infty. \quad (\text{MT}_{ue}^{\mathbb{H}})$$

The power $n/(n-1)$ in the denominator is sharp in the sense that the supremum above will become infinite if we replace the power $n/(n-1)$ in the denominator by any $p < n/(n-1)$. It is evidence that $(\text{MT}_{ue}^{\mathbb{H}})$ implies $(\text{MT}_{us}^{\mathbb{H}})$ and $(\text{MT}_{uc}^{\mathbb{H}})$.

1.2.2. Adams inequalities on \mathbb{H}^n . Moser–Trudinger-type inequality of higher order derivatives, or Adams-type inequality, in hyperbolic spaces was recently established in [FM15, KS16]. In [KS16], Karmakar and Sandeep proved a sharp Adams-type inequality in \mathbb{H}^n with even n under the condition

$$\int_{\mathbb{H}^n} (P_{n/2} u) u dV_g \leq 1,$$

where P_k denotes the $(2k)$ th order GJMS operator defined by

$$\begin{cases} P_1 = -\Delta_g - n(n-2)/4, \\ P_k = P_1(P_1 + 2)(P_1 + 6) \cdots (P_1 + k(k-1)). \end{cases}$$

More precisely, they established the following inequality

$$\sup_{\substack{u \in C_0^\infty(\mathbb{H}^n): \\ \int_{\mathbb{H}^n} (P_{n/2} u) u dV_g \leq 1}} \int_{\mathbb{H}^n} (e^{\beta(n,n/2)u^2} - 1) dV_g < +\infty. \quad (\text{A}^{\mathbb{H}2})$$

The constant $\beta(n, n/2)$ in $(\text{A}^{\mathbb{H}2})$ is sharp and cannot be improved.

For any integer m less than n , let us denote

$$\nabla_g^m = \begin{cases} \Delta_g^{m/2} & \text{if } k \text{ even,} \\ \nabla_g \Delta_g^{(m-1)/2} & \text{if } k \text{ odd.} \end{cases}$$

In [FM15], Fontana and Morpurgo established the following sharp Adams inequality in the entire hyperbolic space \mathbb{H}^n as follows

$$\sup_{\substack{u \in C_c^\infty(\mathbb{H}^n): \\ \|\nabla_g^m u\|_{n/m} \leq 1}} \int_{\mathbb{H}^n} \Phi_{n,m}(\beta(n, m) |u|^{n/(n-m)}) dV_g < +\infty. \quad (\text{A}_u^{\mathbb{H}})$$

The constant $\beta(n, m)$ is sharp in the sense that the supremum above will become infinite if we replace $\beta(n, m)$ by any $\beta > \beta(n, m)$.

Motivated by $(\text{A}_e^{\mathbb{R}})$, in the recent paper [Kar15], Karmakar established a sharp Adams-type inequality in \mathbb{H}^4 with the exact growth condition as follows

$$\sup_{\substack{u \in W^{2,2}(\mathbb{H}^4) \setminus \{0\}: \\ \int_{\mathbb{H}^4} P_2(u) u dV_g \leq 1}} \frac{1}{\|u\|_2^2} \int_{\mathbb{H}^4} \frac{e^{32\pi^2 u^2} - 1}{(1 + |u|)^2} dV_g < +\infty. \quad (\text{A}_{ue}^{\mathbb{H}})$$

Moreover, this inequality is optimal in the sense that the supremum above will become infinite if the power 2 in the denominator is replaced by any $p < 2$.

1.3. Main results. As can be seen, the sharp Adams-type inequality with exact growth condition for general $n \geq 3$ remains open. In the first part of this paper, we will provide a sharp Adams-type inequality with exact growth condition in \mathbb{H}^n for all $n \geq 3$ under the norm $\|\Delta_g u\|_{n/2}$. Our first main result is stated as follows.

Theorem 1.1. *There exists a constant $C_n > 0$ such that for all $u \in W^{2,n/2}(\mathbb{H}^n)$, with $\|\Delta_g u\|_{n/2} \leq 1$, there holds*

$$\int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta(n,2)|u|^{n/(n-2)})}{(1+|u|)^{n/(n-2)}} dV_g \leq C_n \|u\|_{n/2}^{n/2}. \quad (\text{AMT}_{ue}^{\mathbb{H}})$$

Moreover, this inequality is optimal in the sense that if we consider

$$\sup_{\substack{u \in W^{2,n/2}(\mathbb{H}^n) \setminus \{0\}: \\ \|\Delta_g u\|_{n/2} \leq 1}} \frac{1}{\|u\|_{n/2}^{n/2}} \int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta|u|^{n/(n-2)})}{(1+|u|)^p} dV_g,$$

then the supremum above will become infinite either for $\beta > \beta(n,2)$ and $p = n/(n-2)$, or $\beta = \beta(n,2)$ and $p < n/(n-2)$.

As can be easily seen, $(\text{AMT}_{ue}^{\mathbb{H}})$ when $n = 4$ is slightly different from the result of Karmakar $(\text{A}_{ue}^{\mathbb{H}})$. To prove Theorem 1.1, we use the ideas in the proof of $(\text{A}_e^{\mathbb{R}})$ given in [LTZ15, MS14] plus some useful inequalities involving the decreasing rearrangement given in Section 2.

An immediate consequence of Theorem 1.1 is the following subcritical sharp Adams-type inequality in the spirit of Adachi and Tanaka in $W^{2,n/2}(\mathbb{H}^n)$.

Theorem 1.2. *For any $\alpha \in (0, \beta(n,2))$, there exists a constant $C(n, \alpha) > 0$ such that*

$$\int_{\mathbb{H}^n} \Phi_{n,2}(\alpha|u|^{n/(n-2)}) dV_g \leq C(n, \alpha) \|u\|_{n/2}^{n/2} \quad (\text{AMT}_{us}^{\mathbb{H}})$$

for any function $u \in W^{2,n/2}(\mathbb{H}^n)$ with $\|\Delta_g u\|_{n/2} \leq 1$. Inequality $(\text{AMT}_{us}^{\mathbb{H}})$ is sharp in the sense that it is false if $\alpha \geq \beta(n,2)$. Furthermore, we have the following estimate for $C(n, \alpha)$

$$C(n, \alpha) \leq \frac{C(n)}{\beta(n,2) - \alpha}, \quad (1.2)$$

where $C(n)$ depends only on n .

It is worth noting that Theorem 1.2 give an asymptotic behavior of the constant $C(n, \alpha)$ in the subcritical sharp Adams-type inequality when α tends to $\beta(n,2)$. Such a result on the Euclidean space can be found in [LTZ15] for the Moser–Trudinger and Adams inequalities.

In view of Theorem 1.2, it is easy to obtain a critical sharp Adams-type inequality in $W^{2,n/2}(\mathbb{H}^n)$ involving the norm

$$\|u\|_{W^{2,n/2},\tau} = (\|\Delta_g u\|_{n/2}^{n/2} + \tau \|u\|_{n/2}^{n/2})^{2/n}$$

where $\tau > 0$. This is the content of the following result.

Theorem 1.3. *Let $\tau > 0$, there exist a constant $C(n, \tau) > 0$ such that*

$$\sup_{u: \|u\|_{W^{2,n/2},\tau} \leq 1} \int_{\mathbb{H}^n} \Phi_{n,2}(\beta(n,2)|u|^{n/(n-2)}) dV_g \leq C(n, \tau). \quad (\text{AMT}_{uc}^{\mathbb{H}})$$

The constant $\beta(n, 2)$ is sharp in the sense that the supremum above will become infinite if we replace $\beta(n, 2)$ by any $\beta > \beta(n, 2)$. Furthermore, we have the following estimate for $C(n, \tau)$

$$C(n, \tau) \leq C(n)/\tau, \quad (1.3)$$

where $C(n)$ depends only on n .

In the next part of our paper, we also prove that Theorem 1.2 can imply an improved version of the sharp Adams inequality $(\text{AMT}_{uc}^{\mathbb{H}})$ in the spirit of Lions [Lio85]. To make this statement clear, we shall prove the following result.

Theorem 1.4. *There exists a constant $C(n) > 0$ such that for any $u \in W^{2, n/2}(\mathbb{H}^n)$ with $\|\Delta_g u\|_{n/2} < 1$, the following inequality holds*

$$\int_{\mathbb{H}^n} \Phi_{n,2} \left(\frac{2^{2/(n-2)} \beta(n, 2)}{(1 + \|\Delta_g u\|_{n/2}^{n/2})^{2/(n-2)}} |u|^{\frac{n}{n-2}} \right) dV_g \leq C(n) \frac{\|u\|_{n/2}^{n/2}}{1 - \|\Delta_g u\|_{n/2}^{n/2}}. \quad (\text{AMT}_{ucL}^{\mathbb{H}})$$

Consequently, we have for any $\tau > 0$,

$$\sup_{u: \|u\|_{W^{2, n/2, \tau}} \leq 1} \int_{\mathbb{H}^n} \Phi_{n,2} \left(\frac{2^{2/(n-2)} \beta(n, 2)}{(1 + \|\Delta_g u\|_{n/2}^{n/2})^{2/(n-2)}} |u|^{\frac{n}{n-2}} \right) dV_g \leq \frac{C(n)}{\tau}. \quad (1.4)$$

The constant $\beta(n, 2)$ is sharp in the sense that inequality $(\text{AMT}_{ucL}^{\mathbb{H}})$ does not hold if we replace $\beta(n, 2)$ by any larger constant.

Notice that if $\|\Delta_g u\|_{n/2} < 1$, then $2^{2/(n-2)}(1 + \|\Delta_g u\|_{n/2}^{n/2})^{-2/(n-2)} \beta(n, 2) > \beta(n, 2)$. Therefore (1.4) is indeed an improvement of $(\text{AMT}_{uc}^{\mathbb{H}})$. It is noted that the Euclidean versions of $(\text{AMT}_{ucL}^{\mathbb{H}})$ and (1.4) was recently proved by Lam, Lu and Tang in [LLT16, Theorem 1.5]. Their proofs are based on the domain decomposition method. Our proof below is different with theirs and is derived from Theorem 1.2. Despite the fact that Theorem 1.3 can be derived from Theorem 1.2, however, it turns out that these two theorems are in fact equivalent; see Section 4 below. It seems very surprise since Theorem 1.3 is the critical version of the sharp Adams-type inequality while Theorem 1.2 is the subcritical version. In Euclidean spaces, this fact was recently addressed by Lam, Lu and Zhang in [LLZ15]. It is evident that Theorem 1.4 implies Theorem 1.3. Hence, up to a constant $C(n)$ depending only on n , these three theorems are equivalent.

We also establish a sharp Adams–Moser–Trudinger-type inequality in the Sobolev space with homogeneous Navier boundary condition $W_{N,g}^{m,n/m}(\Omega)$ for any bounded domain $\Omega \subset \mathbb{H}^n$, where $W_{N,g}^{m,n/m}(\Omega)$ is defined by

$$W_{N,g}^{m,n/m}(\Omega) = \{u \in W^{m,n/m}(\Omega) : \Delta_g^j u = 0 \text{ on } \partial\Omega, j = 0, 1, \dots, [(m-1)/2]\}.$$

Note that $W_{N,g}^{m,n/m}(\Omega)$ contains the Sobolev space $W_0^{m,n/m}(\Omega)$ as a closed subspace. Our next theorem is a hyperbolic analogue of the result of Tarsi in Euclidean space; see [Tar12, Theorem 4].

Theorem 1.5. *Let $n > 2$, and let Ω be bounded domain in \mathbb{H}^n . There exists a constant $C(n) > 0$ such that for any integer $m \in [1, n)$ and for all $u \in W_{N,g}^{m,n/m}(\Omega)$ with $\|\nabla_g^m u\|_{n/m} \leq 1$, there holds*

$$\int_{\Omega} \exp(\beta(n, m)|u|^{n/(n-m)}) dV_g \leq C(n)|\Omega|. \quad (\text{AMT}_{bcN}^{\mathbb{H}})$$

The constant $\beta(n, m)$ in $(\text{AMT}_{bcN}^{\mathbb{H}})$ is sharp in the sense that the supremum of the left hand side of $(\text{AMT}_{bcN}^{\mathbb{H}})$ in $W_{N,g}^{m,n/m}(\Omega)$ becomes infinity if it is replaced by any larger β .

Another aspect of the Moser–Trudinger and Adams inequalities concerns the concentration–compactness phenomena. In his famous paper [Lio85], Lions proved a so-called concentration–compactness principle for the Moser functional, which asserts that given a bounded domain Ω in \mathbf{R}^n if a sequence $\{u_j\}_j \subset W_0^{1,n}(\Omega)$ with $\|\nabla u_j\|_{L^n(\Omega)} = 1$ converges weakly to a non-zero function $u \in W_0^{1,n}(\Omega)$, then there holds

$$\sup_j \int_{\Omega} \exp(p\beta(n, 1)|u_j|^{n/(n-1)}) dx < +\infty, \quad (1.5)$$

for any $p < (1 - \|\nabla u^\# \|_{L^n(\Omega^\#)}^n)^{-1/(n-1)}$. Here, $u^\#$ and $\Omega^\#$ are rearrangement of u and Ω , respectively; see Section 2 below for the definition. Note that inequality (1.5) does not give any further information than the Moser–Trudinger inequality if the sequence converges weakly to the zero function, but the implication of (1.5) is that the critical Moser functional is compact outside a weak neighborhood of zero function.

In [CCH13], Černý, Cianchi and Hencl improved the Lions result by showing that inequality (1.5) still holds for any

$$p < P_{n,1}(u) =: (1 - \|\nabla u\|_{L^n(\Omega)}^n)^{-1/(n-1)}.$$

Moreover, the threshold $P_{n,1}(u)$ is sharp. A more detailed discussions on Lions lemma and their generalizations to functions with unrestricted boundary condition can be found in [CCH13].

Recently, the Lions lemma for the Moser functional has been extended on whole space \mathbf{R}^n by do Ó, de Souza, de Medeiros and Severo [OM14b] by exploiting the approach of Černý, Cianchi and Hencl [CCH13]. The concentration–compactness principle for the Adams functional has been established by do Ó and Macedo [OM14a] by using the rearrangement argument and the generalization of the Talenti comparison principle. In a very recent paper, Lions-type lemma for the Adams functional on whole space \mathbf{R}^n was proved by Nguyen [Ngu16]. The method using in [Ngu16] is a further modification of the method of Černý, Cianchi and Hencl [CCH13] and is completely based on some estimates for decreasing rearrangement of functions in terms of their higher order derivatives.

Following the approach in [Ngu16], we establish a Lions-type lemma for the Adams inequality in the whole hyperbolic space \mathbb{H}^n . To the best of our knowledge, the Lions-type lemma for the Adams inequality in \mathbb{H}^n is still open except for a few cases. For examples, it was established by Karmakar [Kar15] in $W^{1,n}(\mathbb{H}^n)$ and $W^{2,n/2}(\mathbb{H}^n)$ by using cover lemma and the Lions-type lemma for Moser–Trudinger and Adams inequalities on bounded domain of \mathbf{R}^n . However, his proof is completely different with ours given below. The following is our result.

Theorem 1.6. *Let m be a positive integer less than n and let $\{u_j\}_j$ be a sequence in $W^{m,n/m}(\mathbb{H}^n)$ such that $\|\nabla_g^m u_j\|_{n/m} \leq 1$ and u_j converges weakly to a non-zero function u on $W^{m,n/m}(\mathbb{H}^n)$. Then*

$$\sup_j \int_{\mathbb{H}^n} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)}) dV_g < +\infty \quad (\text{AMT}_{CC}^{\mathbb{H}})$$

for all $p < P_{n,m}(u)$ where

$$P_{n,m}(u) := \begin{cases} (1 - \|\nabla^m u\|_{n/m}^{n/m})^{-m/(n-m)} & \text{if } \|\nabla^m u\|_{n/m} < 1, \\ +\infty & \text{if } \|\nabla^m u\|_{n/m} = 1. \end{cases}$$

Moreover, the threshold $P_{n,m}(u)$ is sharp in the sense that $(\text{AMT}_{CC}^{\mathbb{H}})$ is no longer true if $p \geq P_{n,m}(u)$.

Note that in the special cases $m = 1, 2$, Theorem 1.6 were proved in [Kar15] by using cover lemma and the Lions-type lemma for the Moser–Trudinger and Adams inequalities on bounded domain of \mathbf{R}^n . Our proof below is based on the rearrangement argument, hence is completely different with the one in [Kar15].

The rest of this paper is organized as follows: We recall some facts about the rearrangement in the hyperbolic space, and prove some useful inequalities involving the rearrangement such as the Talenti-type comparison principle and an estimate for the rearrangement function of the unique weak solution of a Dirichlet problem in hyperbolic space in Section 2. In Section 3, we prove Theorem 1.1. Theorems 1.2, 1.3, and 1.4 will be proved in Section 4. Then we proved Theorem 1.5 in Section 5. In Section 6, we prove Theorem 1.6.

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2. PRELIMINARIES

2.1. Rearrangement in hyperbolic spaces. It is now known that the symmetrization argument works well in the setting of hyperbolic spaces. It is not only the key tool in the proof of the classical Moser–Trudinger in \mathbb{H}^n [LT13] but also a key tool in our proof of Theorem 1.1.

Let us now recall some facts about the rearrangement in the hyperbolic spaces. Let the function $f : \mathbb{H}^n \rightarrow \mathbf{R}$ be such that

$$|\{x \in \mathbb{H}^n : |f(x)| > t\}| = \int_{\{x \in \mathbb{H}^n : |f(x)| > t\}} dV_g < +\infty$$

for every $t > 0$. Its *distribution function* is defined by

$$\mu_f(t) = |\{x \in \mathbb{H}^n : |f(x)| > t\}|.$$

Then its *decreasing rearrangement* f^* is defined by

$$f^*(t) = \sup\{s > 0 : \mu_f(s) > t\}.$$

Now, we define $f^\sharp : \mathbb{H}^n \rightarrow \mathbf{R}$ by

$$f^\sharp(x) = f^*(|B(0, d(0, x))|),$$

where $B(0, d(0, x))$ denotes the ball centered at the origin 0 with radius $d(0, x)$ in the hyperbolic space. Then for any continuous increasing function $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ we

have

$$\int_{\mathbb{H}^n} \Phi(|f|) dV_g = \int_{\mathbb{H}^n} \Phi(f^\sharp) dV_g.$$

Moreover, the Hardy–Littlewood inequality implies that

$$\int_{\mathbb{H}^n} |fh| dV_g \leq \int_{\mathbb{H}^n} f^\sharp h^\sharp dV_g,$$

for any function $f, h : \mathbb{H}^n \rightarrow \mathbf{R}$.

Since f^* is non-increasing, the *maximal function* f^{**} of the rearrangement f^* defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$$

for $s \geq 0$ is also non-increasing. Furthermore, it is easy to see that $f^{**} \geq f^*$. Moreover, we have:

Lemma 2.1. *Let $f \in L^p(\mathbb{H}^n)$ with $p \in (1, +\infty)$ and $1/p + 1/p' = 1$, then*

$$\left(\int_0^{+\infty} f^{**}(s)^p ds \right)^{1/p} \leq p' \left(\int_0^{+\infty} f^*(s)^p ds \right)^{1/p}.$$

In particular, if $\text{supp } f \subset \Omega$ where Ω is a domain in \mathbb{H}^n , then

$$\left(\int_0^{|\Omega|} f^{**}(s)^p ds \right)^{1/p} \leq p' \left(\int_0^{|\Omega|} f^*(s)^p ds \right)^{1/p}.$$

Lemma 2.1 above is just an immediate consequence of a well-known result of G.H. Hardy, for interested reader, we refer to [MS14, Proposition 3.1].

2.2. Useful inequalities involving rearrangement. In this subsection, we list some useful facts, which shall be used in the proof of Theorem 1.1, whose proofs will be given in the next subsection. We first prove a comparison principle for solution of a Dirichlet problem, which is similar to the one of Talenti in the Euclidean space [Tal76].

Let $\Omega \subset \mathbb{H}^n$, $n \geq 2$ be a bounded, open set and let $f \in L^2(\Omega)$. We consider the following Dirichlet problem

$$\begin{cases} -\Delta_g u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Let us denote by Ω^\sharp the ball centered at origin such that $|\Omega^\sharp| = |\Omega|$ and consider the Dirichlet problem

$$\begin{cases} -\Delta_g v = f^\sharp & \text{in } \Omega^\sharp, \\ v = 0 & \text{on } \partial\Omega^\sharp. \end{cases} \quad (2.2)$$

Then we have the following comparison principle.

Proposition 2.2. *Let u and v be the solution of (2.1) and (2.2), respectively. Then there holds $u^\sharp \leq v$ in Ω^\sharp .*

We next use Proposition 2.2 to obtain the comparison principle for higher derivatives Δ_g^k ; see Proposition 2.3 below.

Suppose Ω be a bounded open domain in \mathbb{H}^n , for any function $u \in C_0^\infty(\Omega)$, let us denote

$$f = (-\Delta_g)^k u, \quad u_i = (-\Delta_g)^i u$$

for $i = 0, 1, \dots, k-1$. It is obvious to see that u_0, u_1, \dots, u_{k-1} solve the following problems

$$\begin{cases} -\Delta_g u_i = u_{i+1} & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \quad (P_i)$$

for all $i = 0, 1, \dots, k-2$ and

$$\begin{cases} -\Delta_g u_{k-1} = f & \text{in } \Omega \\ u_{k-1} = 0 & \text{on } \partial\Omega. \end{cases} \quad (P_{k-1})$$

We also consider similar problems on Ω^\sharp given by

$$\begin{cases} -\Delta_g v_{k-1} = f^\sharp & \text{in } \Omega^\sharp \\ v_{k-1} = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \quad (P_{k-1}^\sharp)$$

and

$$\begin{cases} -\Delta_g v_i = v_{i+1} & \text{in } \Omega^\sharp \\ v_i = 0 & \text{on } \partial\Omega^\sharp \end{cases} \quad (P_i^\sharp)$$

for $i = 0, 1, \dots, k-2$. Then we have the following result.

Proposition 2.3. *For any $i = 0, 1, \dots, k-1$, we have*

$$u_i^\sharp(x) \leq v_i(x)$$

everywhere in Ω^\sharp .

We also establish the following estimate for the rearrangement function of solutions of (2.1) which is a hyperbolic analogue of [MS14, Proposition 3.4] and is a crucial tool in the proof of Theorem 1.1.

Proposition 2.4. *Given $f \in L^{n/2}(\Omega)$, let u be the unique weak solution of (2.1). Then*

$$u^*(t_1) - u^*(t_2) \leq \frac{1}{(n\Omega_n^{1/n})^2} \int_{t_1}^{t_2} \frac{f^{**}(s)}{s^{1-2/n}} ds,$$

for any $0 < t_1 < t_2 \leq |\Omega|$.

To prove Theorem 1.5, we need an estimate for the arrangement function of solutions of problems (P_i) with $i = 0, 1, \dots, k-1$, which is a higher order derivative version of Proposition 2.4. To be precise, we will prove the following result.

Proposition 2.5. *Let $f \in L^{n/(2k)}(\Omega)$ and let u be weak solution of problems (P_i) with $i = 0, 1, \dots, k-1$. Then there holds*

$$u^*(t) \leq \frac{n}{n-2k} \frac{c_{n,k}}{(n\Omega_n^{1/n})^{2k}} \int_t^{|\Omega|} \frac{f^*(s)}{s^{1-\frac{2k}{n}}} ds + \frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k}} t^{\frac{2k}{n}-1} \int_0^t f^*(s) ds$$

where

$$c_{n,k} = \begin{cases} \frac{n^{2(k-1)}}{2^{k-1}(k-1)! \prod_{j=1}^{k-1} (n-2j)} & \text{if } k \geq 2, \\ 1, & \text{if } k = 1. \end{cases}$$

As mentioned before, the rest of this section is devoted to prove Propositions 2.2, 2.3, 2.4, and 2.5.

2.3. Proofs of Propositions 2.2, 2.3, 2.4, and 2.5. In this subsection, we first prove Proposition 2.2. Then we show that Proposition 2.4 follows from the proof of Proposition 2.2. Proposition 2.3 is proved by applying consecutively Proposition 2.2 and maximum principle. Proposition 2.5 is proved by applying consecutively Proposition 2.4.

Proof of Proposition 2.2. Our proof follows closely the argument in [Tal76]. For fixed $t, h > 0$, by applying the Hölder inequality, we get

$$\frac{1}{h} \int_{\{t < |u| \leq t+h\}} |\nabla_g u| dV_g \leq \left(\frac{1}{h} \int_{\{t < |u| \leq t+h\}} |\nabla_g u|^2 dV_g \right)^{1/2} \left(\frac{\mu_u(t) - \mu_u(t+h)}{h} \right)^{1/2}.$$

Letting $h \downarrow 0$, we obtain

$$-\frac{d}{dt} \int_{\{|u|>t\}} |\nabla_g u|_g dV_g \leq \left(-\frac{d}{dt} \int_{\{|u|>t\}} |\nabla_g u|_g^2 dV_g \right)^{1/2} (-\mu'_u(t))^{1/2}. \quad (2.3)$$

Using the co-area formula, we deduce that

$$\begin{aligned} \int_{\{|u|>t\}} |\nabla_g u|_g dV_g &= \int_{\{|u|>t\}} |\nabla u| \left(\frac{2}{1-|x|^2} \right)^{n-1} dx \\ &= \int_{\{|s|>t\}} \int_{\{u=s\}} \left(\frac{2}{1-|x|^2} \right)^{n-1} d\mathcal{H}^{n-1}(x) ds, \end{aligned}$$

where $d\mathcal{H}^{n-1}(x)$ denotes the $(n-1)$ -dimensional Hausdorff measure. Consequently, for almost everywhere $t > 0$, we obtain

$$-\frac{d}{dt} \int_{\{|u|>t\}} |\nabla_g u|_g dV_g = \int_{\{|u|=t\}} \left(\frac{2}{1-|x|^2} \right)^{n-1} d\mathcal{H}^{n-1}(x).$$

Let $\rho(t)$ denote the radius of the ball centered at origin such that

$$|B(0, \rho(t))| = \mu_u(t), \quad t > 0.$$

Applying the isoperimetric inequality in hyperbolic space [BDS15], we obtain

$$\begin{aligned} \int_{\{|u|=t\}} \left(\frac{2}{1-|x|^2} \right)^{n-1} d\mathcal{H}^{n-1}(x) &\geq \int_{\partial B(0, \rho(t))} \left(\frac{2}{1-|x|^2} \right)^{n-1} d\mathcal{H}^{n-1}(x) \\ &= n\Omega_n \sinh^{n-1} \rho(t). \end{aligned}$$

On the other hand, we have

$$\mu_u(t) = |B(0, \rho(t))| = n\Omega_n \int_0^{\rho(t)} \sinh^{n-1}(s) ds.$$

Hence there exists a continuous, strictly increasing function F such that

$$\rho(t) = F(\mu_u(t)).$$

Consequently, we obtain the following estimate

$$1 \leq \frac{-\mu'_u(t)}{[n\Omega_n \sinh^{n-1} F(\mu_u(t))]^2} \left(-\frac{d}{dt} \int_{\{|u|>t\}} |\nabla_g u|_g^2 dV_g \right). \quad (2.4)$$

For fixed $t, h > 0$, let us define the test function

$$\phi(x) = \begin{cases} 0 & \text{if } |u| \leq t, \\ (|u| - t) \operatorname{sign}(u) & \text{if } t < |u| \leq t+h, \\ h \operatorname{sign}(u) & \text{if } |u| > t+h. \end{cases}$$

Clearly $\phi \in W_0^{1,2}(\Omega)$ and

$$\int_{\Omega} \langle \nabla_g u, \nabla_g \phi \rangle_g dV_g = \int_{\Omega} f \phi dV_g$$

since u is weak solution of (2.1). An easy computation shows that

$$\begin{aligned} \int_{\{t < |u| \leq t+h\}} |\nabla_g u|_g^2 dV_g &= \int_{\Omega} \langle \nabla_g u, \nabla_g \phi \rangle_g dV_g \\ &= \int_{\{t < |u| \leq t+h\}} f(|u| - t) \operatorname{sign}(u) dV_g + \int_{\{|u| > t+h\}} fh \operatorname{sign}(u) dV_g \\ &= \int_{\{t < |u|\}} f(|u| - t) \operatorname{sign}(u) dV_g - \int_{\{t+h < |u|\}} f(|u| - t - h) \operatorname{sign}(u) dV_g. \end{aligned}$$

Dividing both sides by h then letting $h \downarrow 0$ in the resulting equation, and using the Hardy–Littlewood inequality, we obtain

$$\begin{aligned} -\frac{d}{dt} \int_{\{|u| > t\}} |\nabla_g u|_g^2 dV_g &= -\frac{d}{dt} \int_{\{|u| > t\}} f(|u| - t) \operatorname{sign}(u) dV_g = \int_{\{|u| > t\}} f \operatorname{sign}(u) dV_g \\ &\leq \int_{\{|u| > t\}} |f| dV_g \\ &\leq \int_0^{\mu_u(t)} f^*(s) ds = \mu_u(t) f^{**}(\mu_u(t)). \end{aligned} \tag{2.5}$$

Plugging (2.5) into (2.4) and integrating the resulting over (s', s) to get

$$\begin{aligned} s - s' &\leq \int_{s'}^s \frac{-\mu'_u(t)}{[n\Omega_n \sinh^{n-1} F(\mu_u(t))]^2} \mu_u(t) f^{**}(\mu_u(t)) dt \\ &= \int_{\mu_u(s)}^{\mu_u(s')} \frac{1}{[n\Omega_n \sinh^{n-1} F(t)]^2} t f^{**}(t) dt. \end{aligned} \tag{2.6}$$

Letting $s' \rightarrow 0$ in (2.6) we obtain

$$s \leq \int_{\mu_u(s)}^{|\Omega|} \frac{1}{[n\Omega_n \sinh^{n-1} F(t)]^2} t f^{**}(t) dt.$$

For any $t \in (0, |\Omega|)$, if $u^*(t) > 0$, then for any $0 < s < u^*(t)$ we must have $\mu_u(s) > t$ by the definition of the rearrangement function. Therefore

$$s \leq \int_t^{|\Omega|} \frac{1}{[n\Omega_n \sinh^{n-1} F(r)]^2} r f^{**}(r) dr.$$

Letting $s \uparrow u^*(t)$ we get

$$u^*(t) \leq \int_t^{|\Omega|} \frac{1}{[n\Omega_n \sinh^{n-1} F(r)]^2} r f^{**}(r) dr.$$

It is obvious that if $u^*(t) = 0$, then the inequality above is true. Hence for any $t \in (0, |\Omega|)$ we have

$$u^*(t) \leq \int_t^{|\Omega|} \frac{1}{[n\Omega_n \sinh^{n-1} F(r)]^2} r f^{**}(r) dr.$$

It is easy to check that

$$v(x) = \int_{|B(0, d(0, x))|}^{|\Omega|} \frac{1}{[n\Omega_n \sinh^{n-1} F(r)]^2} r f^{**}(r) dr,$$

is unique solution of (2.2). The inequality $u^\sharp \leq v$ obviously holds true, hence the proof of Proposition 2.2 is finished. \square

Proof of Proposition 2.4. Using the simple inequality $\cosh s \geq 1$, it is evident that

$$\begin{aligned} \mu_u(t) = |B(0, \rho(t))| &= n\Omega_n \int_0^{\rho(t)} \sinh^{n-1}(s) ds \\ &\leq n\Omega_n \int_0^{\rho(t)} \sinh^{n-1}(s) \cosh(s) ds \end{aligned}$$

$$= \Omega_n \sinh^n \rho(t) = \Omega_n \sinh^n F(\mu_u(t)).$$

Hence, we obtain

$$\sinh^{n-1} F(r) \geq \left(\frac{r}{\Omega_n}\right)^{1-1/n}. \quad (2.7)$$

Combining (2.7) with (2.6) implies

$$s - s' \leq \frac{1}{[n\Omega_n^{1/n}]^2} \int_{\mu_u(s)}^{\mu_u(s')} \frac{f^{**}(t)}{t^{1-2/n}} dt.$$

Now, let $0 < t_1 < t_2 \leq |\Omega|$, if $u^*(t_1) = u^*(t_2)$, then the conclusion of Proposition 2.4 is trivial. If $u^*(t_1) > u^*(t_2)$, then for any s, s' such that $u^*(t_2) < s' < s < u^*(t_1)$, by the definition of rearrangement function, we obviously have $\mu_u(s) > t_1$ and $\mu_u(s') \leq t_2$. Then we have

$$s - s' \leq \frac{1}{(n\Omega_n^{1/n})^2} \int_{\mu_u(s)}^{\mu_u(s')} \frac{f^{**}(t)}{t^{1-2/n}} dt \leq \frac{1}{(n\Omega_n^{1/n})^2} \int_{t_1}^{t_2} \frac{f^{**}(t)}{t^{1-2/n}} dt.$$

Letting $s \uparrow u^*(t_1)$ and $s' \downarrow u^*(t_2)$ implies our desired inequality. \square

Proof of Proposition 2.3. By Proposition 2.2, we have $u_{k-1}^\# \leq v_{k-1}$. We argue by induction argument. Suppose that for some $1 \leq i < k$, we have $u_{k-i}^\# \leq v_{k-i}$, we will show that $u_{k-i-1}^\# \leq v_{k-i-1}$. Indeed, consider the problem

$$\begin{cases} -\Delta_g \Omega = u_{k-i}^\# & \text{in } \Omega^\# \\ \Omega = 0 & \text{on } \partial\Omega^\#. \end{cases}$$

By Proposition 2.2 we have $u_{k-i-1}^\# \leq \Omega$. Since $u_{k-i}^\# \leq v_{k-i}$, we can apply the maximum principle to get $\Omega \leq v_{k-i-1}$. Therefore $u_{k-i-1}^\# \leq v_{k-i-1}$. This finishes our proof of Proposition 2.3. \square

Proof of Proposition 2.5. If $k = 1$, then by Proposition 2.4 we have

$$u^*(t) \leq \frac{1}{n^2 \Omega_n^{2/n}} \int_t^{|\Omega|} \frac{f^* * (s)}{s^{1-2/n}} ds = \frac{1}{n^2 \Omega_n^{2/n}} \int_t^{|\Omega|} \left(\int_0^s f^*(r) dr \right) s^{2/n-2} ds.$$

Integration by parts then implies our desired estimate. If $k \geq 2$, then by denoting $u_k = f$ we have from Proposition 2.4 that

$$u_i^*(t) \leq \frac{1}{n^2 \Omega_n^{2/n}} \int_t^{|\Omega|} \frac{u_{i+1}^{**}(s)}{s^{1-2/n}} ds$$

for all $i = 0, 1, \dots, k-1$. From this we conclude that

$$u_i^{**}(t) \leq \frac{1}{n^2 \Omega_n^{2/n}} \int_0^{|\Omega|} g(t, s) u_{i+1}^{**}(s) ds,$$

where

$$g(t, s) = \begin{cases} t^{-1} s^{2/n} & \text{if } s < t, \\ s^{2/n-1} & \text{if } s \geq t. \end{cases}$$

We consecutively define a sequence of functions $\{G_j\}_{j \geq 1}$ as follows:

$$\begin{cases} G_1(t, s) = g(t, s), \\ G_i(t, s) = \int_0^{|\Omega|} G_{i-1}(t, s') g(s', s) ds' & \text{for all } i \geq 2. \end{cases}$$

Clearly

$$u^*(t) \leq \frac{1}{(n\Omega_n^{1/n})^{2k}} \int_t^{|\Omega|} r^{2/n-1} \left(\int_0^{|\Omega|} G_{k-1}(r, s) f^{**}(s) ds \right) dr.$$

Choose $R > 0$ such that $R^n \Omega_n = |\Omega|$ and denote by B_R the centered ball of radius R in \mathbf{R}^n . For $x \in B_R$, let us define

$$g(x) = f^*(\Omega_n |x|^n)$$

and

$$v(x) = \frac{1}{(n\Omega_n^{1/n})^{2k}} \int_{\Omega_n |x|^n}^{|\Omega|} r^{2/n-1} \left(\int_0^r G_{k-1}(r, s) f^{**}(s) ds \right) dr.$$

Then the rearrangement function of g , being considered in \mathbf{R}^n , satisfies $g^* = f^*$ and

$$v^*(t) = \frac{1}{(n\Omega_n^{1/n})^{2k}} \int_t^{|\Omega|} r^{2/n-1} \left(\int_0^r G_{k-1}(r, s) f^{**}(s) ds \right) dr.$$

Some straightforward computations show that

$$\begin{cases} (-\Delta)^k v = g & \text{in } B_R \\ (-\Delta)^i v = 0 & \text{on } \partial B_R \quad \text{for all } i = 0, 1, \dots, k-1. \end{cases}$$

We extend $g = 0$ outside the ball B_R and define

$$w(x) = \frac{c_{n,k}}{(n-2k)n^{2k-1}\Omega_n} \int_{\mathbf{R}^n} |x-y|^{2k-n} g(y) dy.$$

It is easy to verify that

$$\begin{cases} (-\Delta)^k w = g & \text{in } B_R \\ (-\Delta)^i w = 0 & \text{on } \partial B_R \quad \text{for all } i = 0, 1, \dots, k-1. \end{cases}$$

In view of the maximum principle, we obtain $v(x) \leq w(x)$ for $x \in B_R$. Equivalently, there holds $v^*(t) \leq w^*(t)$ for any $t \in (0, |\Omega|)$. On the other hand, it follows from a result due to O'Neil [ONe63] that

$$\begin{aligned} w^*(t) &\leq \frac{c_{n,k}}{(n-2k)n^{2k-1}\Omega_n} \left(\frac{1}{t} \int_0^t \left(\frac{\Omega_n}{s} \right)^{1-2k/n} ds \int_0^t g^*(s) ds + \int_t^{+\infty} g^*(s) \left(\frac{\Omega_n}{s} \right)^{1-2k/n} ds \right) \\ &= \frac{n}{n-2k} \frac{c_{n,k}}{(n\Omega_n^{1/n})^{2k}} \int_t^{|\Omega|} f^*(s) s^{2k/n-1} ds + \frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k}} t^{\frac{2k}{n}-1} \int_0^t f^*(s) ds, \end{aligned} \quad (2.8)$$

since $g^*(s) = f^*(s)$ and $g^*(s) = 0$ for $s > |\Omega|$. From this, Proposition 2.5 follows from (2.8) and the estimates $u^* \leq v^* \leq w^*$. \square

3. ADAMS INEQUALITY WITH EXACT GROWTH: PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 by the same way in [LTZ15, MS14]. In the following subsection, we introduce some crucial tools which shall be used in our proof.

3.1. Some crucial lemmas.

Lemma 3.1. *Let $p > 1$ and let $u, f \in L^p((0, +\infty))$ be decreasing functions such that*

$$u(t_1) - u(t_2) \leq c \int_{t_1}^{t_2} \frac{f(s)}{s^{1-1/p}} ds, \quad (3.1)$$

for any $0 < t_1 < t_2$ and c is a positive constant. If $u(A) > 1$, and

$$\int_A^{+\infty} f(s)^p ds \leq \left(\frac{p}{p-1} \right)^p,$$

then

$$\frac{\exp\left(\left(\frac{p-1}{cp}\right)^{p/(p-1)} u(A)^{p/(p-1)}\right)}{(u(A))^{p/(p-1)}} A \lesssim \int_A^{+\infty} u(s)^p ds.$$

This lemma is proved by using Proposition 2.4 and the following lemma whose proof can be found in [LTZ15].

Lemma 3.2. *Given any sequence $a = (a_k)_{k \geq 0}$ and any $p > 1$ let us denote*

$$\|a\|_1 = \sum_{k=0}^{+\infty} |a_k|, \quad \|a\|_p = \left(\sum_{k=0}^{+\infty} |a_k|^p \right)^{1/p}, \quad \|a\|_{(e)} = \left(\sum_{k=0}^{+\infty} |a_k|^p e^k \right)^{1/p},$$

and

$$\mu(h) = \inf \{ \|a\|_{(e)} : \|a\|_1 = h, \|a\|_p \leq 1 \}.$$

Then we have

$$\mu(h) \sim \exp(h^{p/(p-1)}/p) h^{-1/(p-1)}$$

for $h > 1$

Proof of Lemma 3.1. Denote $h_k = c_1 u(e^k A)$, where $c_1 = (p-1)/cp$. Define $a_k = h_k - h_{k+1} \geq 0$, hence

$$\sum_{k=0}^{+\infty} |a_k| = h_0 = c_1 u(A).$$

It follows from (3.1) and Hölder inequality that

$$a_k = c_1 (u(e^k A) - u(e^{k+1} A)) \leq \frac{p-1}{p} \left(\int_{e^k A}^{e^{k+1} A} f(s)^p ds \right)^{1/p}.$$

Consequently, we have

$$\sum_{k=0}^{+\infty} |a_k|^p \leq \left(\frac{p-1}{p} \right)^p \int_A^{+\infty} f(s)^p ds \leq 1.$$

On the other hand

$$\begin{aligned} \frac{1}{A} \int_A^{+\infty} u(s)^p ds &= \sum_{k=0}^{+\infty} \frac{1}{A} \int_{e^k A}^{e^{k+1} A} u(s)^p ds \\ &\geq \sum_{k=0}^{+\infty} u(e^{k+1} A)^p e^k (e-1) \\ &\geq \frac{e-1}{e} \sum_{k=1}^{+\infty} a_k^p e^k. \end{aligned}$$

Therefore,

$$\|a\|_{(e)}^p = a_0^p + \sum_{k=1}^{+\infty} a_k^p e^k \lesssim h_0^p + \frac{1}{A} \int_A^{+\infty} u(s)^p ds. \quad (3.2)$$

Next we estimate h_0 . To do this, we choose $b = (c_1/2)^{p/(p-1)}$; hence for any $1 \leq r \leq e^b$, we have

$$\begin{aligned} h_0 - c_1 u(rA) &\leq \frac{p-1}{p} \int_A^{e^b A} \frac{f(s)}{s^{1-1/p}} ds \\ &\leq \frac{p-1}{p} \left(\int_A^{e^b A} f(s)^p ds \right)^{1/p} b^{1-1/p} \\ &\leq \frac{c_1}{2} \leq \frac{h_0}{2}, \end{aligned}$$

here we use the inequality $u(A) > 1$. From this, we have $h_0 \leq 2c_1 u(rA)$ for any $1 \leq r \leq e^b$. Therefore,

$$\frac{1}{A} \int_A^{+\infty} u(s)^p ds \geq \frac{1}{A} \int_A^{e^b A} u(s)^p ds \geq \left(\frac{h_0}{2c_1} \right)^p (e^b - 1) \gtrsim h_0^p. \quad (3.3)$$

Combining (3.3) and (3.2) gives

$$\|a\|_{(e)}^p \lesssim \frac{1}{A} \int_A^{+\infty} u(s)^p ds.$$

By Lemma 3.2, we obtain

$$\|a\|_{(e)}^p \gtrsim h_0^{-p/(p-1)} \exp(h_0^{p/(p-1)}) \gtrsim (u(A))^{-p/(p-1)} \exp((c_1 u(A))^{p/(p-1)}).$$

This completes the proof of Lemma 3.1. \square

We next recall a lemma due to Adams which plays a crucial role in [Ada88].

Lemma 3.3. *Let $p > 1$ and $p' = p/(p-1)$, and let $a(s, t)$ be a nonnegative measurable function on $\mathbf{R} \times [0, +\infty)$ such that $a(s, t) \leq 1$ for $0 < s < t$, and*

$$\sup_{t>0} \left(\int_{-\infty}^0 + \int_t^{+\infty} a(s, t)^{p'} ds \right)^{1/p'} = b < +\infty.$$

Then there exists a constant $c_0 = c_0(p, b)$ such that if for $\phi \geq 0$, $\int_{\mathbf{R}} \phi(t)^p dt \leq 1$, then we have

$$\int_0^{+\infty} e^{-F(t)} dt < c_0,$$

where

$$F(t) = t - \left(\int_{\mathbf{R}} a(s, t) \phi(s) ds \right)^{p'}.$$

3.2. Proof of Theorem 1.1. We are now in a position to prove Theorem 1.1. For the sake of simplicity, we divide our proof into several steps.

3.2.1. Proof of (AMT_{ue}^H). Using density argument, we only need to prove Theorem 1.1 for all functions $u \in C_0^\infty(\mathbb{H}^n)$. By the property of rearrangement, we have

$$\int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta(n, 2)|u|^{n/(n-2)})}{(1 + |u|)^{n/(n-2)}} dV_g = \int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g,$$

and

$$\|u\|_{n/2}^{n/2} = \|u^\sharp\|_{n/2}^{n/2}.$$

Therefore, it suffices to prove that

$$\int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \leq C \|u^\sharp\|_{n/2}^{n/2}. \quad (3.4)$$

Then we will split the integral appearing in (3.4) into two parts as done in [LTZ15, MS14].

$$\begin{aligned} & \int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \\ &= \left(\int_{B(0, R_0)} + \int_{\mathbb{H}^n \setminus B(0, R_0)} \right) \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g, \end{aligned}$$

where

$$R_0 = \inf\{r \geq 0 : u^*(|B(0, r)|) \leq 1\} \in [0, +\infty).$$

Next we estimate the two integrals term by term. To estimate the integral $\int_{\mathbb{H}^n \setminus B(0, R_0)}$, we observe that

$$\begin{cases} u^*(|B(0, r)|) > 1 & \text{when } r < R_0, \\ u^*(|B(0, R_0)|) = 1, \\ u^*(|B(0, r)|) \leq 1 & \text{when } r > R_0. \end{cases}$$

Since

$$\Phi_{n,2}(\beta(n, 2)x^{n/(n-2)}) \leq Cx^{n/2}$$

for $0 \leq x \leq 1$, we conclude that

$$\begin{aligned} & \int_{\mathbb{H}^n \setminus B(0, R_0)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \\ & \leq C \int_{\mathbb{H}^n \setminus B(0, R_0)} (u^\sharp)^{n/2} dV_g \leq C \|u\|_{n/2}^{n/2}. \end{aligned} \quad (3.5)$$

We next consider the integral $\int_{B(0, R_0)}$. For simplicity, we denote

$$f = -\Delta_g u$$

in \mathbb{H}^n and

$$\alpha = \int_0^{+\infty} (f^{**}(s))^{n/2} ds.$$

Then by Lemma 2.1, we obtain

$$\alpha \leq \left(\frac{n}{n-2}\right)^{n/2} \int_0^{+\infty} [f^*(s)]^{n/2} ds = \left(\frac{n}{n-2}\right)^{n/2} \|\Delta_g u\|_{n/2}^{n/2} \leq \left(\frac{n}{n-2}\right)^{n/2}.$$

Fix $0 < \epsilon_0 < 1$ and let us define R_1 be such that

$$\int_0^{|B(0, R_1)|} [f^{**}(s)]^{n/2} ds \leq \alpha \epsilon_0, \quad \int_{|B(0, R_1)|}^{+\infty} [f^{**}(s)]^{n/2} ds \leq \alpha(1 - \epsilon_0).$$

By applying Proposition 2.4 and the Hölder inequality, we have

$$u^*(t_1) - u^*(t_2) \leq \frac{1}{(n\Omega_n^{1/n})^2} \left(\int_{t_1}^{t_2} [f^{**}(s)]^{n/2} ds \right)^{2/n} \left(\ln \frac{t_2}{t_1} \right)^{1-2/n},$$

for any $0 < t_1 < t_2$. Then

$$u^*(|B(0, r_1)|) - u^*(|B(0, r_2)|) \leq \frac{(\alpha \epsilon_0)^{2/n}}{(n\Omega_n^{1/n})^2} \left(\ln \frac{|B(0, r_2)|}{|B(0, r_1)|} \right)^{1-2/n}, \quad (3.6)$$

for any $0 < r_1 < r_2 < R_1$ and

$$u^*(|B(0, r_1)|) - u^*(|B(0, r_2)|) \leq \frac{(\alpha(1 - \epsilon_0))^{2/n}}{(n\Omega_n^{1/n})^2} \left(\ln \frac{|B(0, r_2)|}{|B(0, r_1)|} \right)^{1-2/n}, \quad (3.7)$$

for any $r_2 > r_1 > R_1$. In order to estimate the integral on $B(0, R_0)$, we need to consider the two cases: $R_1 \geq R_0$ and $R_1 < R_0$.

Case 1: Suppose $R_1 \geq R_0$. By (3.6), we obtain

$$u^*(|B(0, r)|) \leq 1 + \frac{(\alpha \epsilon_0)^{2/n}}{(n\Omega_n^{1/n})^2} \left(\ln \frac{|B(0, R_0)|}{|B(0, r)|} \right)^{1-2/n}.$$

for any $0 < r \leq R_0$. Using the elementary inequality

$$(1 + s^{(n-2)/n})^{n/(n-2)} \leq (1 + \epsilon)s + C_\epsilon$$

for $s \geq 0$ with $C_\epsilon = (1 - (1 + \epsilon)^{-(n-2)/2})^{-2/(n-2)}$, we obtain

$$[u^*(|B(0, r)|)]^{n/(n-2)} \leq (1 + \epsilon) \frac{(\alpha \epsilon_0)^{2/(n-2)}}{(n\Omega_n^{1/n})^{2n/(n-2)}} \ln \frac{|B(0, R_0)|}{|B(0, r)|} + C_\epsilon.$$

Choosing $\epsilon = 1 - \epsilon_0^{2/(n-2)}$, then $(1 + \epsilon)\epsilon_0^{2/(n-2)} < 1$. Since $\alpha \leq (n/(n-2))^{n/2}$ and

$$\beta(n, 2)(n\Omega_n^{1/n})^{-2n/(n-2)} = ((n-2)/n)^{n/(n-2)},$$

we know that

$$\begin{aligned} & \int_{B(0, R_0)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \\ & \leq \int_{B(0, R_0)} \exp(\beta(n, 2)|u^\sharp|^{n/(n-2)}) dV_g \end{aligned}$$

$$\begin{aligned}
&= n\Omega_n \int_0^{R_0} \exp(\beta(n, 2)|u^*(|B(0, r)|)|^{n/(n-2)}) \sinh^{n-1}(r) dr \\
&\leq e^{\beta(n, 2)C_\epsilon} n\Omega_n \int_0^{R_0} \exp\left((1+\epsilon)\epsilon_0^{2/(n-2)} \ln \frac{|B(0, R_0)|}{|B(0, r)|}\right) \sinh^{n-1}(r) dr \\
&= e^{\beta(n, 2)C_\epsilon} |B(0, R_0)|^{(1+\epsilon)\epsilon_0^{2/(n-2)}} \int_0^{|B(0, R_0)|} s^{-(1+\epsilon)\epsilon_0^{2/(n-2)}} ds \\
&\lesssim |B(0, R_0)| \\
&\lesssim n\Omega_n \int_0^{R_0} u^*(|B(0, r)|)^{n/2} \sinh^{n-1}(r) dr \\
&\lesssim \|u\|_{n/2}^{n/2}.
\end{aligned} \tag{3.8}$$

From this we get the desired inequality when $R_1 \geq R_0$, thanks to (3.5) and (3.8).

Case 2: Suppose $R_1 < R_0$. We split the integral $\int_{B(0, R_0)}$ into two parts as follows

$$\int_{B(0, R_0)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g = \left(\int_{B(0, R_0) \setminus B(0, R_1)} + \int_{B(0, R_1)} \right) \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g.$$

The estimate of the integral on $B(0, R_0) \setminus B(0, R_1)$ is much easier. In fact, by inequality (3.7), we have

$$u^*(|B(0, r)|) \leq 1 + \frac{(\alpha(1 - \epsilon_0))^{2/n}}{(n\Omega_n^{1/n})^2} \left(\ln \frac{|B(0, R_0)|}{|B(0, r)|} \right)^{1-2/n}$$

for any $R_1 < r < R_0$. Denote $\epsilon_1 = 1 - (1 - \epsilon_0)^{2/(n-2)}$, then $(1 + \epsilon_1)(1 - \epsilon_0)^{2/(n-2)} < 1$. Similar to Case 1 above, we have

$$u^*(|B(0, r)|)^{n/(n-2)} \leq (1 + \epsilon_1) \frac{(\alpha(1 - \epsilon_0))^{2/(n-2)}}{(n\Omega_n^{1/n})^{2n/(n-2)}} \ln \frac{|B(0, R_0)|}{|B(0, r)|} + C_{\epsilon_1}.$$

Hence

$$\begin{aligned}
&\int_{B(0, R_0) \setminus B(0, R_1)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \\
&\leq \int_{B(0, R_0) \setminus B(0, R_1)} \exp(\beta(n, 2)[u^\sharp]^{n/(n-2)}) dV_g \\
&\leq n\Omega_n \int_{R_1}^{R_0} \exp(\beta(n, 2)u^*(|B(0, r)|)^{n/(n-2)}) \sinh^{n-1}(r) dr \\
&\lesssim n\Omega_n \int_{R_1}^{R_0} \left(\frac{|B(0, R_0)|}{|B(0, r)|} \right)^{(1+\epsilon_1)(1-\epsilon_0)^{2/(n-2)}} \sinh^{n-1}(r) dr \\
&\lesssim |B(0, R_0)|^{(1+\epsilon_1)(1-\epsilon_0)^{2/(n-2)}} \int_{|B(0, R_1)|}^{|B(0, R_0)|} s^{-(1+\epsilon_1)(1-\epsilon_0)^{2/(n-2)}} ds \\
&\lesssim |B(0, R_0)| \\
&\lesssim n\Omega_n \int_0^{R_0} u^*(|B(0, r)|)^{n/2} \sinh^{n-1}(r) dr \\
&\lesssim \|u\|_{n/2}^{n/2}.
\end{aligned} \tag{3.9}$$

Next we estimate the integral on $B(0, R_1)$. Note that when $0 < r < R_1$

$$u^*(|B(0, r)|) = [u^*(|B(0, r)|) - u^*(|B(0, R_1)|)] + u^*(|B(0, R_1)|).$$

Hence by Proposition 2.4 we get

$$u^*(|B(0, r)|)^{n/(n-2)} \leq (1 + \epsilon_2)[u^*(|B(0, r)|) - u^*(|B(0, R_1)|)]^{n/(n-2)} + C_{\epsilon_2}[u^*(|B(0, R_1)|)]^{n/(n-2)}$$

$$\leq (1 + \epsilon_2) \left(\frac{1}{(n\Omega_n^{1/n})^2} \int_{|B(0,r)|}^{B(0,R_1)} \frac{f^{**}(s)}{s^{1-2/n}} ds \right)^{n/(n-2)} + C_{\epsilon_2} [u^*(|B(0, R_1)|)]^{n/(n-2)}.$$

Recall that $\beta(n, 2) = (n\Omega_n^{1/n})^{2n/(n-2)}((n-2)/n)^{n/(n-2)}$. Therefore,

$$\begin{aligned} & \int_{B(0,R_1)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \\ & \lesssim \frac{n\Omega_n}{[u^*(|B(0, R_1)|)]^{n/(n-2)}} \int_0^{R_1} \exp\left(\beta(n, 2)u^*(|B(0, r)|)^{n/(n-2)}\right) \sinh^{n-1}(r) dr \\ & \lesssim \frac{n\Omega_n \exp\left(C_{\epsilon_2}[u^*(|B(0, R_1)|)]^{n/(n-2)}\right)}{[u^*(|B(0, R_1)|)]^{n/(n-2)}} \\ & \quad \times \int_0^{R_1} \exp\left(\left[\frac{(1 + \epsilon_2)^{(n-2)/n} \frac{n-2}{n}}{\times \int_{|B(0,r)|}^{B(0,R_1)} \frac{f^{**}(s)}{s^{1-2/n}} ds} \right]^{n/(n-2)}\right) \sinh^{n-1}(r) dr \\ & = \frac{\exp\left(C_{\epsilon_2}[u^*(|B(0, R_1)|)]^{n/(n-2)}\right)}{[u^*(|B(0, R_1)|)]^{n/(n-2)}} \\ & \quad \times \int_0^{|B(0,R_1)|} \exp\left(\left[\frac{(1 + \epsilon_2)^{(n-2)/n} \frac{n-2}{n}}{\times \int_r^{|B(0,R_1)|} \frac{f^{**}(s)}{s^{1-2/n}} ds} \right]^{n/(n-2)}\right) dr. \end{aligned}$$

Using the change of variables $r = e^{-t}|B(0, R_1)|$, we have

$$\begin{aligned} & \int_{B(0,R_1)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \\ & \lesssim |B(0, R_1)| \frac{\exp\left(C_{\epsilon_2}[u^*(|B(0, R_1)|)]^{n/(n-2)}\right)}{[u^*(|B(0, R_1)|)]^{n/(n-2)}} \\ & \quad \times \int_0^{+\infty} \exp\left(\left[\frac{(1 + \epsilon_2)^{(n-2)/n} \frac{n-2}{n}}{\times \int_{e^{-t}|B(0,R_1)|}^{|B(0,R_1)|} \frac{f^{**}(s)}{s^{1-2/n}} ds} \right]^{n/(n-2)}\right) e^{-t} dt \\ & \lesssim |B(0, R_1)| \frac{\exp\left(C_{\epsilon_2}\beta(n, 2)[u^*(|B(0, R_1)|)]^{n/(n-2)}\right)}{[u^*(|B(0, R_1)|)]^{n/(n-2)}} \\ & \quad \times \int_0^{+\infty} \exp\left(\left[\frac{|B(0, R_1)|^{2/n} (1 + \epsilon_2)^{(n-2)/n} \frac{n-2}{n}}{\times \int_0^t f^{**}(|B(0, R_1)|e^{-s})e^{-\frac{2s}{n}} ds} \right]^{n/(n-2)}\right) e^{-t} dt. \end{aligned} \tag{3.10}$$

Let us now define

$$\varphi(t) = |B(0, R_1)|^{2/n} (1 + \epsilon_2)^{(n-2)/n} \frac{n-2}{n} f^{**}(|B(0, R_1)|e^{-t}) e^{-2t/n} \chi_{\{t>0\}}.$$

Then by the choice of R_1 , we get

$$\begin{aligned} \int_{\mathbf{R}} \varphi(t)^{n/2} dt &= |B(0, R_1)| (1 + \epsilon_2)^{(n-2)/n} \left(\frac{n-2}{n}\right)^{n/2} \int_0^{+\infty} f^{**}(|B(0, R_1)|e^{-t})^{n/2} e^{-t} dt \\ &= (1 + \epsilon_2)^{(n-2)/n} \left(\frac{n-2}{n}\right)^{n/2} \int_0^{|B(0,R_1)|} [f^{**}(s)]^{n/2} ds \\ &\leq \epsilon_0 (1 + \epsilon_2)^{(n-2)/n}. \end{aligned}$$

By choosing $\epsilon_2 = \epsilon_0^{-2/(n-2)} - 1$, we then have $\int_{\mathbf{R}} \varphi(t)^{n/2} dt \leq 1$. Setting $a(s, t) = \chi_{(0, t)}(s)$. By (3.10) and Lemma 3.3, we have

$$\int_{B(0, R_1)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \lesssim |B(0, R_1)| \frac{\exp\left(C_{\epsilon_2}\beta(n, 2)[u^*(|B(0, R_1)|)]^{n/(n-2)}\right)}{[u^*(|B(0, R_1)|)]^{n/(n-2)}}.$$

Note that $C_{\epsilon_2} = (1 - \epsilon_0)^{-2/(n-2)}$, therefore

$$\int_{B(0, R_1)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \lesssim |B(0, R_1)| \frac{\exp\left((1 - \epsilon_0)^{-2/(n-2)}\beta(n, 2)[u^*(|B(0, R_1)|)]^{n/(n-2)}\right)}{[u^*(|B(0, R_1)|)]^{n/(n-2)}}$$

Recall that

$$\int_{|B(0, R_1)|}^{+\infty} [f^{**}(s)]^{n/2} ds \leq (n/(n-2))^{n/2} (1 - \epsilon_0).$$

Applying Lemma 3.1 to the functions $u^*(1 - \epsilon)^{-2/n}$, $f^{**}(1 - \epsilon)^{-2/n}$, $p = n/2$, $c = (n\Omega_n^{1/n})^{-2}$, and $A = |B(0, R_1)|$, we then have

$$\begin{aligned} & |B(0, R_1)| \exp\left((1 - \epsilon_0)^{-2/(n-2)}\beta(n, 2)[u^*(|B(0, R_1)|)]^{n/(n-2)}\right) \\ & \times [u^*(|B(0, R_1)|)]^{-n/(n-2)} \lesssim (1 - \epsilon_0)^{-n/(n-2)} \int_{|B(0, R_1)|}^{+\infty} [u^*(s)]^{n/2} ds \lesssim \|u\|_{n/2}^{n/2}. \end{aligned}$$

Therefore

$$\int_{B(0, R_1)} \frac{\Phi_{n,2}(\beta(n, 2)|u^\sharp|^{n/(n-2)})}{(1 + |u^\sharp|)^{n/(n-2)}} dV_g \lesssim \|u\|_{n/2}^{n/2}. \quad (3.11)$$

Combining (3.11) and (3.9) finishes our proof of Case 2, and hence completes our proof of inequality (AMT_{ue}^H).

3.2.2. *The sharpness of (AMT_{ue}^H).* It remains to prove the sharpness of Theorem 1.1. To see this, let us consider the sequence of functions

$$v_m(x) = \begin{cases} \left(\frac{1}{\beta(n, 2)} \ln m\right)^{1-2/n} + \frac{n}{2}\beta(n, 2)^{2/n-1} \frac{1-m^{2/n}|x|^2}{(\ln m)^{2/n}} & \text{if } 0 \leq |x| \leq m^{-1/n}, \\ -n\beta(n, 2)^{2/n-1} (\ln m)^{-2/n} \ln |x| & \text{if } m^{-1/n} \leq |x| \leq 1, \\ \xi_m(x) & \text{if } |x| > 1, \end{cases}$$

where $\xi_m \in C_0^\infty(\mathbf{R}^n)$ is a radial function such that $\xi_m(x) = 0$ if $|x| \geq 2$, and

$$\xi_m|_{\{|x|=1\}} = 0, \quad \frac{\partial \xi_m}{\partial r}|_{\{|x|=1\}} = -n\beta(n, 2)^{2/n-1} (\ln m)^{-2/n},$$

and ξ_m , $|\nabla \xi_m|$ and $\Delta \xi_m$ are all $O((\ln m)^{-2/n})$. The choice of this sequence is inspired by the sequence in [MS14] in 4-dimensional Euclidean space case, and by the sequence in [LTZ15] in general dimensional Euclidean space.

Following the idea in [Kar15], let us define $\tilde{v}_m(x) = v_m(3x)$, then $\tilde{v}_m \in W^{2, n/2}(\mathbb{H}^n)$ for all m . Moreover, we can readily check that

$$\|\tilde{v}_m\|_{n/2}^{n/2} = O\left(\frac{1}{\ln m}\right)$$

and

$$1 \leq \|\Delta_g \tilde{v}_m\|_{n/2}^{n/2} \leq 1 + O\left(\frac{1}{\ln m}\right).$$

Setting $u_m = \tilde{v}_m \|\Delta_g \tilde{v}_m\|_{n/2}^{-1}$, we obtain $\|u_m\|_{n/2}^{n/2} \leq O(1/\ln m)$ and $\|\Delta_g u_m\|_{n/2}^{n/2} = 1$. Moreover, for any $\beta > 0$ and $p > 0$, we have

$$\int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta u_m^{n/(n-2)})}{(1 + |u|)^p} dV_g \geq \int_{\{|x| \leq 3^{-1}m^{-1/n}\}} \frac{\Phi_{n,2}(\beta u_m^{n/(n-2)})}{(1 + |u|)^p} dV_g$$

$$\begin{aligned}
&\gtrsim (\ln m)^{-(n-2)p/n} \int_{\{|x| \leq 3^{-1}m^{-1/n}\}} \exp\left(\frac{\beta}{\beta(n, 2)} \frac{\ln m}{\left(1 + O\left(\frac{1}{\ln m}\right)\right)^{n/(n-2)}}\right) dV_g \\
&\gtrsim (\ln m)^{-(n-2)p/n} \int_0^{3^{-1}m^{-1/n}} \exp\left(\frac{\beta}{\beta(n, 2)} \ln m\right) r^{n-1} dr \\
&\geq (\ln m)^{-(n-2)p/n} \exp\left(\left(\frac{\beta}{\beta(n, 2)} - 1\right) \ln m\right).
\end{aligned}$$

Therefore, we get

$$\frac{1}{\|u_m\|_{n/2}^{n/2}} \int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta u_m^{n/(n-2)})}{(1+|u|)^p} dV_g \gtrsim (\ln m)^{1-(n-2)p/n} \exp\left(\left(\frac{\beta}{\beta(n, 2)} - 1\right) \ln m\right).$$

This shows that if $\beta > \beta(n, 2)$ or $\beta = \beta(n, 2)$ and $p < n/(n-2)$, then

$$\lim_{m \rightarrow +\infty} \frac{1}{\|u_m\|_{n/2}^{n/2}} \int_{\mathbb{H}^n} \frac{\Phi_{n,2}(\beta u_m^{n/(n-2)})}{(1+|u|)^p} dV_g = +\infty.$$

This proves the sharpness of Theorem 1.1.

4. ADAMS-TYPE INEQUALITIES: PROOF OF THEOREMS 1.2, 1.3 AND 1.4

4.1. Proof of Theorem 1.2. Theorem 1.2 is an immediate consequence of Theorem 1.1. Indeed, for any $u \in W^{2,n/2}(\mathbb{H}^n)$ such that $\|\Delta_g u\|_{n/2} \leq 1$, let us denote

$$\Omega = \{x \in \mathbb{H}^n : |u(x)| > 1\}.$$

In Ω^c , we have $|u| \leq 1$, then by the definition of $\Phi_{n,2}$ we have

$$\begin{aligned}
\Phi_{n,2}(\alpha|u|^{n/(n-2)}) &= \sum_{j=j_{n/2}-1}^{+\infty} \frac{\alpha^j}{j!} |u|^{jn/(n-2)} \\
&\leq |u|^{n/2} \sum_{j=j_{n/2}-1}^{+\infty} \frac{\alpha^j}{j!} \\
&\leq e^\alpha |u|^{n/2} \leq e^{\beta(n,2)} |u|^{n/2}.
\end{aligned}$$

(Note that $(j_{n/2} - 1)n/(n-2) \geq n/2$.) Therefore,

$$\int_{\Omega^c} \Phi_{n,2}(\alpha|u|^{n/(n-2)}) dV_g \leq e^{\beta(n,2)} \|u\|_{n/2}^{n/2} \leq \frac{C(n)}{\beta(n, 2) - \alpha} \|u\|_{n/2}^{n/2}. \quad (4.1)$$

In Ω we have $|u| \geq 1$, then

$$\frac{\Phi_{n,2}(\beta(n, 2)|u|^{n/(n-2)})}{(1+|u|)^{n/(n-2)}} \gtrsim \exp(\beta(n, 2)|u|^{n/(n-2)}) |u|^{-n/(n-2)}.$$

Using the elementary inequality $e^{-t} \leq e^{-1}t^{-1}$ for any $t > 0$ and Theorem 1.1, we have

$$\begin{aligned}
\int_{\Omega} \Phi_{n,2}(\alpha|u|^{n/(n-2)}) dV_g &\leq \int_{\Omega} \exp(\alpha|u|^{n/(n-2)}) dV_g \\
&= \int_{\Omega} \exp(\beta(n, 2)|u|^{n/(n-2)}) \exp(-(\beta(n, 2) - \alpha)|u|^{n/(n-2)}) dV_g \\
&\leq \frac{1}{e(\beta(n, 2) - \alpha)} \int_{\Omega} \exp(\beta(n, 2)|u|^{n/(n-2)}) |u|^{-n/(n-2)} dV_g \\
&\lesssim \frac{1}{e(\beta(n, 2) - \alpha)} \int_{\Omega} \frac{\Phi_{n,2}(\beta(n, 2)|u|^{n/(n-2)})}{(1+|u|)^{n/(n-2)}} dV_g \\
&\lesssim \frac{C(n)}{\beta(n, 2) - \alpha} \|u\|_{n/2}^{n/2}. \quad (4.2)
\end{aligned}$$

Combining estimates (4.1) and (4.2), we obtain (AMT $_{us}^{\mathbb{H}}$).

The sharpness of constant $\beta(n, 2)$ follows from Theorem 1.1. Hence we finish the proof of Theorem 1.2.

4.2. Proof of Theorem 1.3. It is enough to prove inequality (AMT $_{uc}^{\mathbb{H}}$) for $u \in W^{2,n/2}(\mathbb{H}^n)$ such that $\|u\|_{W^{2,n/2},\tau} = 1$, we have

$$\|\Delta_g u\|_{n/2}^{n/2} = 1 - \tau \|u\|_{n/2}^{n/2}.$$

Denote

$$v = u \|\Delta_g u\|_{n/2}^{-1}, \quad \alpha = \beta(n, 2) \|\Delta_g u\|_{n/2}^{n/(n-2)}.$$

Clearly, $\|\Delta_g v\|_{n/2} = 1$. It follows from Theorem 1.2 that

$$\begin{aligned} \int_{\mathbb{H}^n} \Phi_{n,2}(\beta(n, 2) |u|^{n/(n-2)}) dV_g &= \int_{\mathbb{H}^n} \Phi_{n,2}(\alpha |v|^{n/(n-2)}) dV_g \\ &\leq \frac{C(n)}{\beta(n, 2) (1 - \|\Delta_g u\|_{n/2}^{n/(n-2)})} \frac{\|u\|_{n/2}^{n/2}}{\|\Delta_g u\|_{n/2}^{n/2}}. \end{aligned}$$

It is easy to show that for any $t \in (0, 1)$ and any $a \in (0, 2]$ there holds

$$(1 - t)^a \leq 1 - \min\{a, 1\} t.$$

Using this elementary inequality, we obtain

$$1 - \|\Delta_g u\|_{n/2}^{n/(n-2)} = 1 - (1 - \tau \|u\|_{n/2}^{n/2})^{2/(n-2)} \geq C(n) \tau \|u\|_{n/2}^{n/2}.$$

Hence if $\|\Delta_g u\|_{n/2} \geq 1/2$, we have

$$\int_{\mathbb{H}^n} \Phi_{n,2}(\beta(n, 2) |u|^{n/(n-2)}) dV_g \leq \frac{C(n)}{\tau}.$$

If $0 < \|\Delta_g u\|_{n/2} \leq 1/2$, then we let $v = 2u$. Clearly, $\|\Delta_g v\|_{n/2} \leq 1$; hence by Theorem 1.2, we have

$$\begin{aligned} \int_{\mathbb{H}^n} \Phi_{n,2}(\beta(n, 2) |u|^{n/(n-2)}) dV_g &= \int_{\mathbb{H}^n} \Phi_{n,2}\left(\frac{\beta(n, 2)}{2^{n/(n-2)}} |v|^{n/(n-2)}\right) dV_g \\ &\leq C(n) \|v\|_{n/2}^{n/2} = C(n) \frac{1 - \|\Delta_g u\|_{n/2}^{n/2}}{\tau} \leq \frac{C(n)}{\tau}. \end{aligned}$$

Therefore, we have shown that

$$\int_{\mathbb{H}^n} \Phi_{n,2}(\beta(n, 2) |u|^{n/(n-2)}) dV_g \leq C(n)/\tau,$$

which is our desired inequality (AMT $_{uc}^{\mathbb{H}}$). To conclude Theorem 1.3, we note that the sharpness of (AMT $_{uc}^{\mathbb{H}}$) follows from the sharpness of (AMT $_{us}^{\mathbb{H}}$) since Theorems 1.2 and 1.3 are equivalent; see Subsection 4.4 below.

4.3. Proof of Theorem 1.4. Fix $u \in W^{2,n/2}(\mathbb{H}^n)$ with $\|\Delta_g u\|_{n/2} < 1$. If $u \equiv 0$, then there is nothing to prove; hence we only consider the case $u \not\equiv 0$. For simplicity, we divide our proof into two cases.

Case 1. Suppose $\|\Delta_g u\|_{n/2} \leq 1/2$. By denoting $v = 2u$, we clearly have,

$$\begin{aligned}
 & \int_{\mathbb{H}^n} \Phi_{n,2} \left(\frac{2^{2/(n-2)} \beta(n, 2)}{(1 + \|\Delta_g u\|_{n/2}^{n/2})^{2/(n-2)}} |u|^{n/(n-2)} \right) dV_g \\
 &= \int_{\mathbb{H}^n} \Phi_{n,2} \left(\frac{\beta(n, 2)}{2(1 + \|\Delta_g u\|_{n/2}^{n/2})^{2/(n-2)}} |v|^{n/(n-2)} \right) dV_g \\
 &\leq \int_{\mathbb{H}^n} \Phi_{n,2} \left(\frac{\beta(n, 2)}{2} |v|^{n/(n-2)} \right) dV_g \\
 &\leq \frac{2C(n)}{\beta(n, 2)} \|v\|_{n/2}^{n/2} \\
 &\leq C(n) \frac{\|u\|_{n/2}^{n/2}}{1 - \|\Delta_g u\|_{n/2}^{n/2}},
 \end{aligned} \tag{4.3}$$

here we have used Theorem 1.2.

Case 2. Suppose $\|\Delta_g u\|_{n/2} \geq 1/2$. In this scenario, let us first denote

$$v = u \|\Delta_g u\|_{n/2}^{-1}, \quad \alpha = \left(\frac{2\|\Delta_g u\|_{n/2}^{n/2}}{1 + \|\Delta_g u\|_{n/2}^{n/2}} \right)^{2/(n-2)} \beta(n, 2).$$

Then it is clear to see that $\|v\|_{n/2} = 1$ and $\alpha < \beta(n, 2)$. By applying Theorem 1.2 we obtain

$$\begin{aligned}
 \int_{\mathbb{H}^n} \Phi_{n,2} \left(\frac{2^{2/(n-2)} \beta(n, 2)}{(1 + \|\Delta_g u\|_{n/2}^{n/2})^{2/(n-2)}} |u|^{\frac{n}{n-2}} \right) dV_g &= \int_{\mathbb{H}^n} \Phi_{n,2} (\alpha |v|^{n/(n-2)}) dV_g \\
 &\leq \frac{C(n)}{\beta(n, 2)} \left(1 - \left(\frac{2\|\Delta_g u\|_{n/2}^{n/2}}{1 + \|\Delta_g u\|_{n/2}^{n/2}} \right)^{\frac{2}{n-2}} \right)^{-1} \frac{\|u\|_{n/2}^{n/2}}{\|\Delta_g u\|_{n/2}^{n/2}}.
 \end{aligned}$$

Since $1 > \|\Delta_g u\|_{n/2} \geq 1/2$, $2/(n-2) \in (0, 2]$, and

$$\frac{2^{1-n/2}}{1 + 2^{-n/2}} \leq \frac{2\|\Delta_g u\|_{n/2}^{n/2}}{1 + \|\Delta_g u\|_{n/2}^{n/2}} \leq 1,$$

there exists some $C'(n) > 0$ such that

$$\left[1 - \left(\frac{2\|\Delta_g u\|_{n/2}^{n/2}}{1 + \|\Delta_g u\|_{n/2}^{n/2}} \right)^{\frac{2}{n-2}} \right]^{-1} \leq C'(n) \left[1 - \frac{2\|\Delta_g u\|_{n/2}^{n/2}}{1 + \|\Delta_g u\|_{n/2}^{n/2}} \right]^{-1} \leq \frac{2C'(n)}{1 - \|\Delta_g u\|_{n/2}^{n/2}}.$$

Therefore

$$\begin{aligned}
 & \int_{\mathbb{H}^n} \Phi_{n,2} \left(\frac{2^{2/(n-2)} \beta(n, 2)}{(1 + \|\Delta_g u\|_{n/2}^{n/2})^{2/(n-2)}} |u|^{\frac{n}{n-2}} \right) dV_g \\
 &\leq \frac{2^{1+n/2} C(n) C'(n)}{\beta(n, 2)} \frac{\|u\|_{n/2}^{n/2}}{1 - \|\Delta_g u\|_{n/2}^{n/2}} \leq C(n) \frac{\|u\|_{n/2}^{n/2}}{1 - \|\Delta_g u\|_{n/2}^{n/2}}.
 \end{aligned} \tag{4.4}$$

Inequality (AMT_{ucL}^H) now follows from the estimates (4.3) and (4.4) above. Finally, we conclude the sharpness of (AMT_{ucL}^H) . To see this, as we have already observed once that

$$2^{2/(n-2)} (1 + \|\Delta_g u\|_{n/2}^{n/2})^{-2/(n-2)} > 1$$

provided $\|\Delta_g u\|_{n/2} < 1$. Therefore, the sharpness of (AMT_{ucL}^H) follows from the sharpness of (AMT_{uc}^H) . The proof of Theorem 1.4 hence is finished.

4.4. Theorems 1.2 and 1.3 are equivalent. Let us finish this section by showing that Theorems 1.2 and 1.3 are, in fact, equivalent. To realize this interesting fact, we only have to show that Theorem 1.2 can be derived from Theorem 1.3.

For any $\alpha \in (0, \beta(n, 2))$ and any $u \in W^{2,n/2}(\mathbb{H}^n)$ such that $\|\Delta_g u\|_{n/2} \leq 1$, we denote

$$v = \left(\frac{\alpha}{\beta(n, 2)} \right)^{(n-2)/n} u, \quad \tau = \frac{1 - \|\Delta_g v\|_{n/2}^{n/2}}{\|v\|_{n/2}^{n/2}}.$$

Clearly,

$$\tau = \frac{\beta(n, 2)^{n/2-1} - \alpha^{n/2-1} \|\Delta_g u\|_{n/2}^{n/2}}{\alpha^{n/2-1} \|u\|_{n/2}^{n/2}} \geq \frac{\beta(n, 2)^{n/2-1} - \alpha^{n/2-1}}{\alpha^{n/2-1} \|u\|_{n/2}^{n/2}}.$$

Applying Theorem 1.3 gives

$$\begin{aligned} \int_{\mathbb{H}^n} \Phi_{n,2}(\alpha |u|^{n/(n-2)}) dV_g &= \int_{\mathbb{H}^n} \Phi_{n,2}(\beta(n, 2) |v|^{n/(n-2)}) dV_g \\ &\leq \frac{C(n)}{\tau} \\ &\leq C(n) \frac{\alpha^{n/2-1} \|u\|_{n/2}^{n/2}}{\beta(n, 2)^{n/2-1} - \alpha^{n/2-1}}. \end{aligned}$$

It is easy to prove that there is some $C'(n)$ depending only on n such that

$$\frac{\alpha^{n/2-1}}{\beta(n, 2)^{n/2-1} - \alpha^{n/2-1}} \leq \frac{C'(n)}{\beta(n, 2) - \alpha}$$

for all $\alpha \in (0, \beta(n, 2))$. Hence, for any $\alpha \in (0, \beta(n, 2))$ we have

$$\int_{\mathbb{H}^n} \Phi_{n,2}(\alpha |u|^{n/(n-2)}) dV_g \leq \frac{C(n)}{\beta(n, 2) - \alpha} \|u\|_{n/2}^{n/2}$$

which is nothing but $(\text{AMT}_{us}^{\mathbb{H}})$.

5. ADAMS INEQUALITY WITH HOMOGENEOUS NAVIER BOUNDARY: PROOF OF THEOREM 1.5

In this section, we are about to prove Theorem 1.5 whose proof relies on Proposition 2.5 and Lemma 3.3.

5.1. Proof of $(\text{AMT}_{bcN}^{\mathbb{H}})$. For simplicity, we divide the proof into two cases.

Case 1. Suppose that m is even. In this case, we can write $m = 2k$ for some $k \geq 1$. This case is a simple consequence of Lemma 2.5 and Lemma 3.3. Indeed, denoting

$$f = (-\Delta_g)^k u$$

and extending f to be zero outside Ω , it follows from Proposition 2.5 that

$$u^*(t) \leq \frac{n}{n-2k} \frac{c_{n,k}}{(n\Omega_n^{1/n})^{2k}} \int_t^{|\Omega|} \frac{f^*(s)}{s^{1-2k/n}} ds + \frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k}} t^{2k/n-1} \int_0^t f^*(s) ds.$$

Recall that

$$\beta(n, 2k) = \left(\frac{n}{n-2k} \frac{c_{n,k}}{(n\Omega_n^{1/n})^{2k}} \right)^{-n/(n-2k)}.$$

Hence by the Hardy–Littlewood inequality, we have

$$\begin{aligned} \int_{\Omega} \exp(\beta(n, 2k) |u|^{n/(n-2k)}) dV_g \\ \leq \int_0^{|\Omega|} \exp(\beta(n, 2k) (u^*(t))^{n/(n-2k)}) dt \end{aligned}$$

$$\leq \int_0^{|\Omega|} \exp \left[\left(\int_t^{|\Omega|} s^{2k/n-1} f^*(s) ds + \frac{n}{2k} t^{2k/n-1} \int_0^t f^*(s) ds \right)^{n/(n-2k)} \right] dt.$$

Changing the variable $t := |\Omega|e^{-t}$, we then obtain

$$\begin{aligned} & \int_{\Omega} \exp(\beta(n, 2k)|u|^{n/(n-2k)}) dV_g \\ & \leq |\Omega| \int_0^{+\infty} \exp \left[-t + \left(\int_{|\Omega|e^{-t}}^{|\Omega|} s^{2k/n-1} f^*(s) ds + \frac{n}{2k} (|\Omega|e^{-t})^{2k/n-1} \int_0^{|\Omega|e^{-t}} f^*(s) ds \right)^{n/(n-2k)} \right] dt. \end{aligned} \quad (5.1)$$

Denote $\phi(s) = f^*(|\Omega|e^{-s})(|\Omega|e^{-s})^{2k/n}$ and

$$a(s, t) = \begin{cases} 0 & \text{if } s < 0, \\ 1 & \text{if } 0 \leq s < t, \\ ne^{(s-t)(2k/n-1)}/(2k) & \text{if } s \geq t. \end{cases}$$

Then by changing of the variables $s := |\Omega|e^{-s}$ in (5.1), it is straightforward to see that

$$\begin{aligned} & \int_{\Omega} \exp(\beta(n, 2k)|u|^{n/(n-2k)}) dV_g \\ & \leq |\Omega| \int_0^{+\infty} \exp \left[-t + \left(\int_0^{+\infty} a(s, t)\phi(s) ds \right)^{n/(n-2k)} \right] dt. \end{aligned}$$

We can easily verify that

$$\int_{\mathbb{R}} \phi(s)^{n/(2k)} ds = \int_0^{|\Omega|} (f^*(s))^{n/(2k)} ds = \int_{\Omega} |f|^{n/(2k)} dV_g = 1$$

and

$$\sup_{t>0} \left[\left(\int_{-\infty}^0 + \int_t^{+\infty} \right) a(s, t)^{n/(n-2k)} ds \right]^{(n-2k)/n} = \frac{n}{2k}.$$

By Lemma 3.3, therefore there is a constant $C(n, k)$ depending only on n and k such that

$$\int_{\Omega} \exp(\beta(n, 2k)|u|^{n/(n-2k)}) dV_g \leq C(n, k)|\Omega|.$$

Case 2. Suppose that m is odd. In this scenario, we can write $m = 2k + 1$ for some $k \geq 0$. If $k = 0$, then the space $W_{N, g}^{1, n}(\Omega)$ is exactly the space $W_0^{1, n}(\Omega)$. Therefore, the conclusion follows from [LT13, Corollary 1.1]. Hence we need to concentrate to the case $k \geq 1$. Denote

$$f = (-\Delta_g)^k u$$

and extend f to be zero outside Ω , then by Lemma 2.5, we obtain

$$u^*(t) \leq \frac{n}{n-2k} \frac{c_{n, k}}{(n\Omega_n^{1/n})^{2k}} \int_t^{|\Omega|} \frac{f^*(s)}{s^{1-2k/n}} ds + \frac{c_{n, k+1}}{(n\Omega_n^{1/n})^{2k}} t^{2k/n-1} \int_0^t f^*(s) ds. \quad (5.2)$$

Recall that

$$f^\sharp(x) = f^*(|B(0, d(0, x))|), \quad d(0, x) = \ln \left(\frac{1+|x|}{1-|x|} \right).$$

Hence

$$\nabla_g f^\sharp(x) = n\Omega_n \frac{1-|x|^2}{2} (f^*)'(|B(0, d(0, x))|) \left(\frac{2|x|}{1-|x|^2} \right)^{n-1}.$$

Thus we have

$$\int_{\mathbb{H}^n} |\nabla_g f^\sharp|^{n/m} dV_g = (n\Omega_n)^{n/m+1} \int_0^1 |(f^*)'(|B(0, \ln \frac{1+r}{1-r})|)|^{n/m} \left(\frac{2r}{1-r^2} \right)^{(n-1)n/m+n} \frac{dr}{r}.$$

Upon using the change of variables

$$s = |B(0, \ln[(1+r)/(1-r)])|$$

we obtain

$$F(s) = \ln[(1+r)/(1-r)]$$

where F is a continuous, strictly increasing function as in the proof of Proposition 2.2. Resolving this equation gives

$$r = (e^{F(s)} - 1)/(e^{F(s)} + 1).$$

Hence

$$\int_{\mathbb{H}^n} |\nabla_g f^\sharp|^{n/m} dV_g = (n\Omega_n)^{n/m} \int_0^{|\Omega|} |(f^*)'(s)|^{n/m} (\sinh F(s))^{n(n-1)/m} ds.$$

(Note that $ds = n\Omega_n(2r/(1-r^2))^n dr/r$.) Let us define the function

$$\varphi(t) = (n\Omega_n)^{-n/(n-m)} \int_t^{|\Omega|} (\sinh F(s))^{-n(n-1)/(n-m)} ds.$$

Then φ is strictly decreasing and has the following asymptotic behavior: $\varphi(|\Omega|) = 0$ and $\lim_{t \rightarrow 0} \varphi(t) = +\infty$. Let g be an increasing function such that $f^*(s) = g(\varphi(s))$, then it is easy to check that

$$(n\Omega_n)^{n/m} \int_0^{|\Omega|} |(f^*)'(s)|^{n/m} (\sinh F(s))^{n(n-1)/m} ds = \int_0^{+\infty} (g'(s))^{n/m} ds.$$

Observe that $\|\nabla_g f^\sharp\|_{n/m} \leq \|\nabla_g f\|_{n/m} \leq 1$; hence

$$\int_0^{+\infty} (g'(s))^{n/m} ds \leq 1.$$

Denote by $k = (g')^*$ the rearrangement function of g' in $(0, +\infty)$, by the Hardy–Littlewood inequality, we obtain

$$f^*(s) = \int_0^{\varphi(s)} g'(t) dt \leq \int_0^{\varphi(s)} k(t) dt$$

for any $s \in (0, |\Omega|)$. By using integration by parts, we get

$$\begin{aligned} \int_t^{|\Omega|} \frac{f^*(s)}{s^{1-2k/n}} ds &\leq \frac{n}{2k} \int_t^{|\Omega|} \int_0^{\varphi(s)} k(r) dr ds^{2k/n} \\ &= -\frac{n}{2k} t^{2k/n} \int_0^{\varphi(t)} k(s) ds - \frac{n}{2k} \int_t^{|\Omega|} k(\varphi(s)) \varphi'(s) s^{2k/n} ds \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \int_0^t f^*(s) ds &\leq \int_0^t \int_0^{\varphi(s)} k(r) dr ds \\ &= t \int_0^{\varphi(t)} k(s) ds - \int_0^t k(\varphi(s)) \varphi'(s) s ds. \end{aligned} \quad (5.4)$$

(Here we have used $\varphi(|\Omega|) = 0$ and $\lim_{s \rightarrow 0} s \int_0^{\varphi(s)} k(r) dr = 0$.) Upon plugging (5.4) and (5.3) into (5.2), we arrive at

$$\begin{aligned} u^*(t) &\leq -\frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k}} \int_t^{|\Omega|} k(\varphi(s)) \varphi'(s) s^{2k/n} ds \\ &\quad - \frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k}} t^{2k/n-1} \int_0^t k(\varphi(s)) \varphi'(s) s ds. \end{aligned}$$

It follows from the definition of φ and (2.7) that

$$\begin{aligned} -\varphi'(s) &= (n\Omega_n)^{-n/(n-m)} (\sinh F(s))^{-n(n-1)/(n-m)} \\ &\leq (n\Omega_n^{1/n})^{-n/(n-m)} s^{-(n-1)/(n-m)}. \end{aligned}$$

Denote

$$l(s) = k(\varphi(s))(-\varphi'(s))^{m/n},$$

then we have

$$\begin{aligned} u^*(t) &\leq \frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k+1}} \int_t^{|\Omega|} \frac{l(s)}{s^{1-(2k+1)/n}} ds + \frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k+1}} t^{2k/n-1} \int_0^t l(s) s^{1/n} ds \\ &\leq \frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k+1}} \int_t^{|\Omega|} \frac{l(s)}{s^{1-(2k+1)/n}} ds + \frac{c_{n,k+1}}{(n\Omega_n^{1/n})^{2k+1}} t^{2k/n} \int_0^t l(s) ds. \end{aligned} \quad (5.5)$$

(Keep in mind that $m = 2k + 1$.) Now we can repeat the argument in the case when m is even to finish the proof of Theorem 1.5 when m is odd by using Lemma 3.3, estimate (5.5), and the fact

$$\int_0^{|\Omega|} l(s)^{n/m} ds \leq \int_0^{+\infty} k(s)^{n/m} ds \leq 1.$$

To conclude Theorem 1.5, it suffices to establish the sharpness of $(\text{AMT}_{bcN}^{\text{H}})$.

5.2. The sharpness of $(\text{AMT}_{bcN}^{\text{H}})$. The way to see the sharpness of $(\text{AMT}_{bcN}^{\text{H}})$ is as follows: Note that since $W_{N,g}^{m,n/m}(\Omega) \subset W_0^{m,n/m}(\Omega)$, the supremum of the left hand side of $(\text{AMT}_{bcN}^{\text{H}})$ in $W_{N,g}^{m,n/m}(\Omega)$ is greater than that in $W_0^{m,n/m}(\Omega)$. Since (A_b^{R}) is sharp, it follows that $(\text{AMT}_{bcN}^{\text{H}})$ is also sharp.

6. A LIONS-TYPE LEMMA FOR ADAMS INEQUALITY: PROOF OF THEOREM 1.6

In this long section, we provide a proof for Theorem 1.6. To achieve that goal, we borrow some ideas in [CCH13] for the case $m = 1$ and a fine analysis in [Ngu16] for the Euclidean case. Our approach basically consists of two steps: First we reduce the sequence $\{u_j\}_j \subset \mathbb{H}^n$ in Theorem 1.6 to the case of $u_j \in C_0^\infty(\mathbb{H}^n)$; see Proposition 6.5. Then we establish Theorem 1.6 for any sequence $u_j \in C_0^\infty(\mathbb{H}^n)$ by way of contradiction; see Subsection 6.3.

6.1. An useful estimate for rearrangement functions. In this subsection, we prove an useful estimate for rearrangement functions; see Proposition 6.3.

Let $u \in C_0^\infty(\mathbb{H}^n)$, our aim is to estimate $u^*(t_1) - u^*(t_2)$ from above for any $0 < t_1 < t_2 < +\infty$. For simplicity, we denote

$$u_i = (-\Delta)^i u$$

for each $i = 0, 1, \dots, k$ with a convention that $u_0 \equiv u$. Then we have

$$u_i^*(t_1) - u_i^*(t_2) \leq \int_{t_1}^{t_2} \frac{t u_{i+1}^{**}(t)}{(n\Omega_n(\sinh F(t))^{n-1})^2} dt \quad (6.1)$$

for all $i = 0, 1, \dots, k-1$. By sending $t_2 \rightarrow +\infty$ and using $\lim_{t \rightarrow +\infty} u_i^*(t) = 0$, we deduce that

$$u_i^*(t) \leq \int_t^{+\infty} \frac{s u_{i+1}^{**}(s)}{(n\Omega_n(\sinh F(s))^{n-1})^2} ds.$$

Now by integrating by parts, we obtain

$$\begin{aligned} u_i^{**}(t) &= \frac{1}{t} \int_0^t u_i^*(s) ds \\ &\leq \frac{1}{t} \int_0^t \left(\int_s^{+\infty} \frac{a u_{i+1}^{**}(a)}{(n\Omega_n(\sinh F(a))^{n-1})^2} da \right) ds \\ &= \int_t^{+\infty} \frac{s u_{i+1}^{**}(s)}{(n\Omega_n(\sinh F(s))^{n-1})^2} ds + \int_0^t \frac{t^{-1} s^2 u_{i+1}^{**}(s)}{(n\Omega_n(\sinh F(s))^{n-1})^2} ds. \end{aligned}$$

Define

$$G(t, s) = \begin{cases} s(n\Omega_n(\sinh F(s))^{n-1})^{-2} & \text{if } s \geq t, \\ t^{-1}s^2(n\Omega_n(\sinh F(s))^{n-1})^{-2} & \text{if } s < t. \end{cases}$$

Then we have

$$u_i^{**}(t) \leq \int_0^{+\infty} G(t, s)u_{i+1}^{**}(s)ds. \quad (6.2)$$

Combining (6.1) and (6.2) gives

$$u_i^*(t_1) - u_i^*(t_2) \leq \int_{t_1}^{t_2} \frac{t}{(n\Omega_n(\sinh F(t))^{n-1})^2} \int_0^{+\infty} G(t, s)u_{i+1}^{**}(s)dsdt \quad (6.3)$$

for all $i = 0, 1, \dots, k-1$. We now define a sequence $(G_i)_{i \geq 1}$ as follows: Set

$$G_1 = G$$

and

$$G_{i+1}(t, s) = \int_0^{+\infty} G_i(t, s_1)G(s_1, s)ds_1 = \int_0^{+\infty} G(t, s_1)G_i(s_1, s)ds_1.$$

Setting $i = 0$ in (6.3) and using (6.2) repeatedly, then we arrive at

$$\begin{aligned} u^*(t_1) - u^*(t_2) &\leq \int_{t_1}^{t_2} \frac{t}{(n\Omega_n(\sinh F(t))^{n-1})^2} \int_0^{+\infty} G_{k-1}(t, s)u_k^{**}(s)dsdt \\ &= \int_0^{+\infty} u_k^{**}(s) \int_{t_1}^{t_2} \frac{t}{(n\Omega_n(\sinh F(t))^{n-1})^2} G_{k-1}(t, s)dt ds. \end{aligned} \quad (6.4)$$

Let us define consecutively the functions L_i, H_i, K_i for $i = 1, 2, \dots, k-1$ by

$$\begin{aligned} L_1(t) &= \frac{t}{(n\Omega_n(\sinh F(t))^{n-1})^2}, \\ H_i(t) &= \int_0^t L_i(s)ds, \\ K_i(t) &= \int_t^{+\infty} s^{-2} H_i(s)ds, \end{aligned}$$

and

$$L_{i+1}(t) = \frac{t}{(n\Omega_n(\sinh F(t))^{n-1})^2} K_i(t).$$

Using these notations, we can rewrite (6.4) as follows

$$u^*(t_1) - u^*(t_2) \leq \int_0^{+\infty} u_k^{**}(s) \int_{t_1}^{t_2} G_{k-1}(t, s)L_1(t)dt ds. \quad (6.5)$$

For $i < k-1$, using integration by parts, we get

$$\begin{aligned} \int_{t_1}^{t_2} G_{k-i}(t, s)L_i(t)dt &= G_{k-i}(t_2, s)H_i(t_2) - G_{k-i}(t_1, s)H_i(t_1) \\ &\quad + \int_{t_1}^{t_2} H(t)t^{-2} \int_0^t \frac{s_1^2}{(n\Omega_n(\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s)ds_1 dt \\ &= G_{k-i}(t_2, s)H_i(t_2) - G_{k-i}(t_1, s)H_i(t_1) \\ &\quad - K_i(t_2) \int_0^{t_2} \frac{s_1^2}{(n\Omega_n(\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s)ds_1 \\ &\quad + K_i(t_1) \int_0^{t_1} \frac{s_1^2}{(n\Omega_n(\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s)ds_1 \\ &\quad + \int_{t_1}^{t_2} G_{k-i-1}(t, s)L_{i+1}(t)dt. \end{aligned}$$

When $i = k - 1$, we use integration by parts again to obtain

$$\begin{aligned} \int_{t_1}^{t_2} G(t, s) L_{k-1}(t) dt &= G(t_2, s) H_{k-1}(t_2) - G(t_1, s) H_{k-1}(t_1) \\ &\quad + \int_{t_1}^{t_2} \chi_{\{s < t\}} \frac{t^{-2} s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} H_{k-1}(t) dt. \end{aligned}$$

We are now able to estimate $\int_{t_1}^{t_2} G_{k-1}(t, s) L_1(t) dt$ as follows

$$\begin{aligned} \int_{t_1}^{t_2} G_{k-1}(t, s) L_1(t) dt &\leq \sum_{i=1}^{k-1} (G_{k-i}(t_2, s) H_i(t_2) - G_{k-i}(t_1, s) H_i(t_1)) \\ &\quad + \sum_{i=1}^{k-2} K_i(t_1) \int_0^{t_1} \frac{s_1^2}{(n \Omega_n (\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s) ds_1 \\ &\quad - \sum_{i=1}^{k-2} K_i(t_2) \int_0^{t_2} \frac{s_1^2}{(n \Omega_n (\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s) ds_1 \\ &\quad + \int_{t_1}^{t_2} \chi_{\{s < t\}} \frac{t^{-2} s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} H_{k-1}(t) dt. \end{aligned} \tag{6.6}$$

When plugging (6.6) into (6.5), there are terms needed separately attention. First we handle the term involving the last term on the right hand side of (6.6). Clearly

$$\begin{aligned} &\int_0^{+\infty} u_k^{**}(s) \int_{t_1}^{t_2} \chi_{\{s < t\}} \frac{t^{-2} s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} H_{k-1}(t) dt ds \\ &= \int_0^{t_2} u_k^{**}(s) \int_{t_1}^{t_2} \chi_{\{s < t\}} \frac{t^{-2} s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} H_{k-1}(t) dt ds \\ &= (K_{k-1}(t_1) - K_{k-1}(t_2)) \int_0^{t_1} u_k^{**}(s) \frac{s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} ds \\ &\quad + \int_{t_1}^{t_2} u_k^{**}(s) \frac{s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} (K_{k-1}(s) - K_{k-1}(t_2)) ds \\ &= K_{k-1}(t_1) \int_0^{t_1} u_k^{**}(s) \frac{s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} ds \\ &\quad - K_{k-1}(t_2) \int_0^{t_2} u_k^{**}(s) \frac{s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} ds \\ &\quad + \int_{t_1}^{t_2} u_k^{**}(s) \frac{s^2}{(n \Omega_n (\sinh F(s))^{n-1})^2} K_{k-1}(s) ds. \end{aligned} \tag{6.7}$$

To handle the term involving the first term on the right hand side of (6.6), let us first denote

$$\begin{aligned} F(t_1, t_2, s) &= \sum_{i=1}^{k-1} (G_{k-i}(t_2, s) H_i(t_2) - G_{k-i}(t_1, s) H_i(t_1)) \\ &\quad + \sum_{i=1}^{k-2} K_i(t_1) \int_0^{t_1} \frac{s_1^2}{(n \Omega_n (\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s) ds_1 \\ &\quad - \sum_{i=1}^{k-2} K_i(t_2) \int_0^{t_2} \frac{s_1^2}{(n \Omega_n (\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s) ds_1. \end{aligned}$$

Hence, combining (6.5), (6.6), and (6.7) gives

$$\begin{aligned}
u^*(t_1) - u^*(t_2) &\leq \int_{t_1}^{t_2} u_k^{**}(s) F(t_1, t_2, s) ds \\
&\quad + K_{k-1}(t_1) \int_0^{t_1} u_k^{**}(s) \frac{s^2}{(n\Omega_n(\sinh F(s))^{n-1})^2} ds \\
&\quad - K_{k-1}(t_2) \int_0^{t_2} u_k^{**}(s) \frac{s^2}{(n\Omega_n(\sinh F(s))^{n-1})^2} ds \\
&\quad + \int_{t_1}^{t_2} u_k^{**}(s) \frac{s^2}{(n\Omega_n(\sinh F(s))^{n-1})^2} K_{k-1}(s) ds.
\end{aligned} \tag{6.8}$$

Our job has not finished yet. In the following step, we aim to estimate $L_i(t)$, $H_i(t)$, $K_i(t)$, and $\int_0^{+\infty} F(t_1, t_2, s)^{n/(n-2k)} ds$.

First, we prove the following result concerning to $L_i(t)$, $H_i(t)$, and $K_i(t)$.

Proposition 6.1. *For all $i = 1, \dots, k-1$, we have the following claims:*

- (1) *There holds $L_i(t) \leq (n\Omega_n^{1/n})^{-2i} c_{n,i} t^{2i/n-1}$ for all $t > 0$ and*

$$L_i(t) \sim \frac{1}{(n-1)^i(i-1)!} \frac{(\ln t)^{i-1}}{t}$$

as $t \rightarrow +\infty$.

- (2) *There holds $H_i(t) \leq (n\Omega_n^{1/n})^{-2i} c_{n,i} t^{2i/n} n/(2i)$ for all $t > 0$ and*

$$H_i(t) \sim \frac{1}{(n-1)^i i!} (\ln t)^i$$

as $t \rightarrow +\infty$.

- (3) *There holds $K_i(t) \leq (n\Omega_n^{1/n})^{-2i} c_{n,i+1} t^{2i/n-1}$ for all $t > 0$ and*

$$K_i(t) \sim \frac{1}{(n-1)^i i!} \frac{(\ln t)^i}{t}$$

as $t \rightarrow +\infty$.

Proof. This is elementary, simply by induction argument; hence we omit its details. \square

Proposition 6.2. *There exists a constant C depending only on n, k such that*

$$G_i(t, s) \leq \begin{cases} C s^{-1+2i/n} & \text{if } s \geq t, \\ C t^{-1+2(i-1)/n} & \text{if } s < t, \end{cases}$$

for $i = 1, 2, \dots, k-1$.

Proof. To prove, we first observe that $\sinh F(s) \geq (s/\Omega_n)^{1/n}$. Therefore,

$$G_1(t, s) \leq \begin{cases} (n\Omega_n^{1/n})^{-2} s^{-1+2/n} & \text{if } s \geq t, \\ (n\Omega_n^{1/n})^{-2} t^{-1} s^{2/n} & \text{if } s < t. \end{cases}$$

This shows that the conclusion holds for $i = 1$. Using induction argument, we obtain the conclusion; for a detailed explanation, we refer the reader to [Ngu16]. \square

An immediately consequence of Proposition 6.2 is the following estimate

$$\int_0^t \frac{s_1^2}{(n\Omega_n(\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s) ds_1 \leq \begin{cases} C t^{2(k-i)/n} & \text{if } s < t, \\ C t^{1+2/n} s^{-1+2(k-i-1)/n} & \text{if } s > t, \end{cases}$$

for $i = 1, \dots, k-2$, which then implies

$$\left(\int_0^{+\infty} \left(\int_0^t \frac{s_1^2}{(n\Omega_n(\sinh F(s_1))^{n-1})^2} G_{k-i-1}(s_1, s) ds_1 \right)^{n/(n-2k)} ds \right)^{(n-2k)/n} \leq C t^{1-2i/n}$$

for $i = 1, \dots, k-2$. This inequality and Proposition 6.1 give

$$\left(\int_0^{+\infty} F(t_1, t_2, s)^{n/(n-2k)} ds \right)^{(n-2k)/n} \leq C \quad (6.9)$$

for any $0 < t_1 < t_2$ where the constant C depends only on n, k . Moreover, we have

$$\begin{aligned} & \int_0^t u_k^{**}(s) \frac{s^2}{(n\Omega_n(\sinh F(s))^{n-1})^2} ds \\ & \leq \frac{1}{(n\Omega_n^{1/n})^2} \int_0^t u_k^{**}(s) s^{2/n} ds \\ & \leq \frac{1}{(n\Omega_n^{1/n})^2} \left(\int_0^{+\infty} (u_k^{**}(s))^{n/(2k)} ds \right)^{2k/n} \left(\int_0^t s^{2/(n-2k)} ds \right)^{(n-2k)/n} \\ & \leq C \left(\int_0^{+\infty} (u_k^{**}(s))^{n/(2k)} ds \right)^{2k/n} t^{1-2(k-1)/n} \end{aligned}$$

for any $t > 0$, here we have used Lemma 2.1. Combining this inequality and Proposition 6.1, we obtain

$$\int_0^t (\Delta^k u)^{**}(s) \frac{s^2}{(n\Omega_n(\sinh F(s))^{n-1})^2} ds K_{k-1}(t) \leq C \|u_k\|_{n/(2k)}. \quad (6.10)$$

Combining (6.8), (6.9), and (6.10), we arrive at

$$u^*(t_1) - u^*(t_2) \leq \int_{t_1}^{t_2} (\Delta^k u)^{**}(s) \frac{s^2}{(n\Omega_n(\sinh F(s))^{n-1})^2} K_{k-1}(s) ds + C \|\Delta^k u\|_{n/(2k)}, \quad (6.11)$$

for any $0 < t_1 < t_2 < +\infty$, with the notation $K_0(s) = s^{-1}$. Denote

$$M(t) = \int_t^{+\infty} \frac{s}{(n\Omega_n(\sinh F(s))^{n-1})^2} K_{k-1}(s) ds.$$

Then we have from Proposition 6.1 that

$$M(t) \leq \frac{c_{n,k}}{(n\Omega_n^{1/n})^{2k}} \frac{n}{n-2k} t^{2k/n-1} \quad (6.12)$$

for all $t > 0$ and

$$M(t) \sim \frac{1}{(n-1)^k(k-1)!} \frac{(\ln t)^{k-1}}{t} \quad (6.13)$$

as $t \rightarrow +\infty$. Using integration by parts, we have

$$\begin{aligned} & \int_{t_1}^{t_2} (\Delta^k u)^{**}(s) \frac{s^2}{(n\Omega_n(\sinh F(s))^{n-1})^2} K_{k-1}(s) ds \\ & = -M(t_2) \int_0^{t_2} (\Delta^k u)^*(s) ds + M(t_1) \int_0^{t_1} (\Delta^k u)^*(s) ds \\ & \quad + \int_{t_1}^{t_2} (\Delta^k u)^*(s) M(s) ds. \end{aligned} \quad (6.14)$$

In view of (6.12), we easily see that

$$M(t) \int_0^t (\Delta^k u)^*(s) ds \leq C \|\Delta^k u\|_{n/(2k)} \quad (6.15)$$

for all $t > 0$ where the constant C depends only on n and k . By combining (6.11), (6.14), and (6.15), we have shown the following key result.

Proposition 6.3. *For any $u \in C_0^\infty(\mathbb{H}^n)$ and for any $1 \leq k < n/2$, there exists a constant $C(n, k)$ such that*

$$u^*(t_1) - u^*(t_2) \leq \int_{t_1}^{t_2} (\Delta^k u)^*(s) M(s) ds + C(n, k) \|\Delta^k u\|_{n/(2k)} \quad (6.16)$$

for any $0 < t_1 < t_2 < +\infty$.

6.2. Reduce to compactly supported smooth functions. We start this section by showing that if $u \in W^{m, n/m}(\mathbb{H}^n)$, then $|u|^{n/(n-m)}$ will be exponential integrability. More precisely, we shall prove:

Lemma 6.4. *For any function $u \in W^{m, n/m}(\mathbb{H}^n)$ and any $p > 0$, we have*

$$\int_{\mathbb{H}^n} \Phi_{n,m}(p|u|^{n/(n-m)}) dV_g < +\infty.$$

Proof. For $\epsilon > 0$, by density, we can choose $v \in C_0^\infty(\mathbb{H}^n)$ such that $\|\nabla^m(u - v)\|_{n/m} < \epsilon$. For any $\delta > 0$, we recall the following elementary inequality

$$|u|^{n/(n-m)} \leq (1 + \delta)|u - v|^{n/(n-m)} + C_\delta |v|^{n/(n-m)}$$

with the constant $C_\delta = (1 - (1 + \delta)^{-(n-m)/m})^{-m/(n-m)}$. Let us divide \mathbb{H}^n into two parts as follows

$$\Omega_1 = \{x : |u(x) - v(x)| \leq 1\},$$

$$\Omega_2 = \{x : |u(x) - v(x)| > 1\}.$$

On Ω_1 , we have $|u| \leq 1 + \max_{\mathbb{H}^n} |v| =: C_v$ then

$$\Phi_{n,m}(p|u|^{n/(n-m)}) \leq C(n, m, p, v)|u|^{n/m}$$

for some constant $C(n, m, p, v) > 0$ depending only on n, m, p , and v . Hence

$$\begin{aligned} \int_{\Omega_1} \Phi_{n,m}(p|u|^{n/(n-m)}) dV_g &\leq \int_{\{|u| \leq C_v\}} \Phi_{n,m}(p|u|^{n/(n-m)}) dV_g \\ &\leq C(n, m, p, v) \int_{\{|u| \leq C_v\}} |u|^{n/m} dV_g \\ &\leq C(n, m, p, v) C \|\nabla^m u\|_{n/m} < +\infty. \end{aligned}$$

(Here we have used the Poincaré–Sobolev inequality in \mathbb{H}^n ; see [FM15, Theorem 18].) For the integral on Ω_2 , we have

$$\begin{aligned} \int_{\Omega_2} \Phi_{n,m}(p|u|^{n/(n-m)}) dV_g &\leq \int_{\Omega_2} \exp(p|u|^{n/(n-m)}) dV_g \\ &\leq \int_{\Omega_2} \exp(2p|u - v|^{n/(n-m)} + pC_1|v|^{n/(n-m)}) dV_g \\ &\leq C(n, m, p, v) \int_{\mathbb{H}^n} \Phi_{n,m}(2p|u - v|^{n/(n-m)}) dV_g, \end{aligned}$$

here we have used the fact that $|u - v| \geq 1$ on Ω_2 and that v is bounded. Choosing ϵ small enough such that $2p\epsilon^{n/(n-m)} \leq \beta(n, m)$, then we have, by the Adams inequality ($\mathbf{A}_u^{\mathbb{H}}$), that

$$\int_{\Omega_2} \Phi_{n,m}(2p|u - v|^{n/(n-m)}) dV_g \leq \int_{\mathbb{H}^n} \Phi_{n,m}(\beta(n, m)(|u - v|/\epsilon)^{n/(n-m)}) dV_g < +\infty,$$

since $\|\nabla^m(u - v)\|_{n/(n-m)} < \epsilon$. Therefore, we obtain

$$\int_{\Omega_2} \Phi_{n,m}(p|u|^{n/(n-m)}) dV_g < +\infty.$$

Thus, we have just shown that

$$\int_{\mathbb{H}^n} \Phi_{n,m}(p|u|^{n/(n-m)})dV_g = \int_{\Omega_1} \Phi_{n,m}(p|u|^{n/(n-m)})dV_g + \int_{\Omega_2} \Phi_{n,m}(p|u|^{n/(n-m)})dV_g < +\infty$$

as claimed. \square

In the following result, we show that it is enough to prove Theorem 1.6 for compactly supported smooth functions.

Proposition 6.5. *Let $\{u_j\}_j$ be a sequence given in Theorem 1.6. Let $v_j \in C_0^\infty(\mathbb{H}^n)$ such that $\|\nabla^m(v_j - u_j)\|_{n/m} < j^{-1}$ for any $j \in \mathbb{N}$. Then for any $p_1 \in (p, P_{n,m}(u))$ there exists constant C such that*

$$\sup_{j \in \mathbb{N}} \int_{\mathbb{H}^n} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \leq C \sup_{j \in \mathbb{N}} \int_{\mathbb{H}^n} \Phi_{n,m}(p_1\beta(n, m)|v_j|^{n/(n-m)})dV_g + C.$$

Proof. It is easy to see that for any $A > 0$, there is a constant $C(n, m, A)$ depending only on n, m, A such that

$$\phi_{n,m}(t) \leq C_{n,m,A} t^{(n-m)/m}$$

for any $t \leq A$. This implies the existence of some constant C , which is independent of j , such that

$$\int_{\{|u_j| \leq 2\}} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \leq C \int_{\mathbb{H}^n} |u_j|^{n/m} dV_g \leq C. \quad (6.17)$$

(Note that we have used the Poincaré inequality once.) We divide the set $\{|u_j| > 2\}$ into two parts as follows

$$\Omega_{j,1} = \{|u_j| > 2\} \cap \{|u_j - v_j| \leq 1\},$$

$$\Omega_{j,2} = \{|u_j| > 2\} \cap \{|u_j - v_j| > 1\}.$$

On $\Omega_{j,1}$ we have $|v_j| \geq |u_j| - |u_j - v_j| > 1$; hence

$$\begin{aligned} & \int_{\Omega_{j,1}} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \\ & \leq \int_{\Omega_{j,1}} \exp(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \\ & \leq \int_{\Omega_{j,1}} \exp(p\beta(n, m)(1 + \delta)|v_j|^{n/(n-m)} + p\beta(n, m)C_\delta)dV_g \quad (6.18) \\ & \leq C \int_{\mathbb{H}^n} \Phi_{n,m}(p_1\beta(n, m)|v_j|^{n/(n-m)})dV_g \\ & \leq C \sup_{j \in \mathbb{N}} \int_{\mathbb{H}^n} \Phi_{n,m}(p_1\beta(n, m)|v_j|^{n/(n-m)})dV_g, \end{aligned}$$

thanks to the fact that $|v_j| \geq 1$ on $\Omega_{j,1}$ and by the choice $\delta = p_1/p - 1$.

On $\Omega_{j,2}$, first we divide it into two parts as follows

$$\Omega_{j,2}^1 = \Omega_{j,2} \cap \{|v_j| < 1\},$$

$$\Omega_{j,2}^2 = \Omega_{j,2} \cap \{|v_j| \geq 1\}.$$

On $\Omega_{j,2}^1$ we have

$$\begin{aligned} & \int_{\Omega_{j,2}^1} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \leq \int_{\Omega_{j,2}^1} \exp(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \\ & \leq \int_{\Omega_{j,2}^1} \exp(2p\beta(n, m)|u_j - v_j|^{n/(n-m)} + p\beta(n, m)C_1)dV_g \end{aligned}$$

$$\leq C \int_{\mathbb{H}^n} \Phi_{n,m}(2p\beta(n, m)|u_j - v_j|^{n/(n-m)})dV_g.$$

Choose J_0 such that $J_0 \geq (2p)^{(n-m)/n}$. Then for any $j \geq J_0$ we have

$$\begin{aligned} & \int_{\mathbb{H}^n} \Phi_{n,m}(2p\beta(n, m)|u_j - v_j|^{n/(n-m)})dV_g \\ & \leq \int_{\mathbb{H}^n} \Phi_{n,m}(\beta(n, m) \left| \frac{u_j - v_j}{\|\nabla^m(u_j - v_j)\|_{n/m}} \right|^{n/(n-m)})dV_g \leq C. \end{aligned}$$

The above estimates and Lemma 6.4 imply that

$$\sup_{j \in \mathbb{N}} \int_{\Omega_{j,2}^1} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \leq C. \quad (6.19)$$

On $\Omega_{j,2}^2$, we have

$$|u_j|^{n/(n-m)} \leq (1 + \epsilon)|v_j|^{n/(n-m)} + C_\epsilon|u_j - v_j|^{n/(n-m)}$$

with $\epsilon = (p_1 - p)/(2p)$. Denote $r = 2p_1/(p + p_1)$ and $r' = r/(r - 1)$. Using the Hölder inequality, we have

$$\begin{aligned} & \int_{\Omega_{j,2}^2} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \\ & \leq \int_{\Omega_{j,2}^2} \exp(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \\ & \leq \int_{\Omega_{j,2}^2} \exp((1 + \epsilon)p\beta(n, m)|v_j|^{n/(n-m)} + p\beta(n, m)C_\epsilon|u_j - v_j|^{n/(n-m)})dV_g \\ & \leq \left(\int_{\Omega_{j,2}^2} \exp(p_1\beta(n, m)|v_j|^{n/(n-m)})dV_g \right)^{(p_1+p)/(2p_1)} \\ & \quad \times \left(\int_{\Omega_{j,2}^2} \exp(r'pC_\epsilon\beta(n, m)|u_j - v_j|^{n/(n-m)})dV_g \right)^{1/r'} \\ & \leq C \left(\int_{\mathbb{H}^n} \Phi_{n,m}(p_1\beta(n, m)|v_j|^{n/(n-m)})dV_g \right)^{(p_1+p)/(2p_1)} \\ & \quad \times \left(\int_{\mathbb{H}^n} \Phi_{n,m}(r'pC_\epsilon\beta(n, m)|u_j - v_j|^{n/(n-m)})dV_g \right)^{1/r'}. \end{aligned}$$

Choosing J_0 such that $J_0 \geq (r'pC_\epsilon)^{(n-m)/n}$ and using the Adams inequality as above, then we have

$$\int_{\mathbb{H}^n} \Phi_{n,m}(r'pC_\epsilon\beta(n, m)|u_j - v_j|^{n/(n-m)})dV_g \leq C$$

for any $j \geq J_0$. Using Lemma 6.4 we obtain

$$\sup_{j \in \mathbb{N}} \int_{\Omega_{j,2}^2} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g \leq C \sup_{j \in \mathbb{N}} \int_{\mathbb{H}^n} \Phi_{n,m}(p_1\beta(n, m)|v_j|^{n/(n-m)})dV_g. \quad (6.20)$$

Combining (6.17), (6.18), (6.19), and (6.20) we obtain the desired result. \square

6.3. Proof of Theorem 1.6 for compactly supported smooth functions. We follow the argument given in [CCH13] by Černý, Cianchi and Hencl. This method was used in [OM14b] to establish the concentration–compactness principle for the Moser–Trudinger inequality in whole space \mathbf{R}^n . Recently, it was used and developed in [Ngu16] to prove the concentration–compactness principle for the sharp Adams–Moser–Trudinger inequality in \mathbf{R}^n for any domain (bounded and unbounded). In the case of bounded domains, the result in [Ngu16] covers the results in [OM14a] for the even order of gradient, and improves the results in [OM14a] for the odd order of gradient.

Let us go back to the proof of Theorem 1.6. As usual, we argue by contradiction. Suppose that there exists a sequence $\{u_j\}_j \subset C_0^\infty(\mathbb{H}^n)$ such that:

- $\|\nabla^m u_j\|_{n/m} \leq 1$,
- u_j converges weakly to a nonzero function u on $W^{m,n/m}(\mathbb{H}^n)$, and
- there exists a number $p \in (1, P_{n,m}(u))$ such that

$$\lim_{j \rightarrow +\infty} \int_{\mathbb{H}^n} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g = +\infty. \quad (6.21)$$

Our aim is to look for a contradiction to (6.21). Using the Rellich–Kondrachov theorem, by passing to a subsequence if necessary, we can assume that:

- u_j converges almost everywhere to u on \mathbb{H}^n ,
- u_j converges to u on $L_{\text{loc}}^p(\mathbb{H}^n)$ for any $p < +\infty$ and additionally
- $\Delta^{(m-1)/2} u_j$ converges almost everywhere to $\Delta^{(m-1)/2} u$ on \mathbb{H}^n if m is odd.

We will need the following simple result.

Lemma 6.6. *Let $u \in L^{n/m}(\mathbb{H}^n)$ such that $\|u\|_{n/m} \leq C$, then for any $R > 0$, $p > 0$ there exists a constant $C(n, m, p, C)$ depending only on n, m, p, C such that*

$$\int_{\{d(0, x) \geq R\}} \Phi_{n,m}(p\beta(n, m)|u^\sharp|^{n/(n-m)})dV_g \leq C(n, m, p, R).$$

Proof. For any $x \in \mathbb{H}^n$, we have

$$\begin{aligned} C^{n/m} &\geq \int_{\mathbb{H}^n} |u|^{n/m} dV_g = \int_{\mathbb{H}^n} |u^\sharp|^{n/m} dV_g \\ &\geq \int_{\{d(0, y) \leq d(0, x)\}} |u^\sharp|^{n/m} dV_g \\ &\geq (u^\sharp(x))^{n/m} |B(0, d(0, x))|, \end{aligned}$$

or equivalently

$$u^\sharp(x) \leq \frac{C}{|B(0, d(0, x))|^{m/n}}.$$

Hence for any y such that $d(0, y) \geq R$, we have $u^\sharp(y) \leq C(n, m, R)$ for some $C(n, m, R)$ depending only on n, m , and R . By the definition of the function $\Phi_{n,m}$, it is easy to check that there exists a constant $C(n, m, p, R)$ depending only on n, m, p , and R such that

$$\Phi_{n,m}(p\beta(n, m)(u^\sharp(y))^{n/(n-m)}) \leq C(n, m, p, R)(u^\sharp(y))^{n/m}$$

for any $d(0, y) \geq R$. This proves Lemma 6.6. \square

We now continue to prove Theorem 1.6. Thanks to $\|\nabla^m u_j\|_{n/m} \leq 1$, we can apply the (rough) Poincaré–Sobolev inequality to obtain $\|u_j\|_{n/m} \leq C$ for any j for some constant $C > 0$ independent of j ; see [FM15]. Now we write

$$\begin{aligned} \int_{\mathbb{H}^n} \Phi_{n,m}(p\beta(n, m)|u_j|^{n/(n-m)})dV_g &= \int_{\mathbb{H}^n} \Phi_{n,m}(p\beta(n, m)|u_j^\sharp|^{n/(n-m)})dV_g \\ &= \left(\int_{B_R} + \int_{B_R^c} \right) \Phi_{n,m}(p\beta(n, m)|u_j^\sharp|^{n/(n-m)})dV_g, \end{aligned}$$

where $B_R = \{x : d(0, x) \leq R\}$ and $B_R^c = \mathbb{H}^n \setminus B_R$. Now Lemma 6.6 and our assumption (6.21) imply that

$$\lim_{j \rightarrow +\infty} \int_{B_R} \Phi_{n,m}(p\beta(n, m)|u_j^\sharp|^{n/(n-m)})dV_g = +\infty. \quad (6.22)$$

Note that for $l < n/m - 1$ by the Hölder inequality we have

$$\int_{B_R} (u^\sharp)^{ln/(n-m)} dV_g \leq \left(\int_{B_R} (u^\sharp)^{n/m} dV_g \right)^{ml/(n-m)} |B_R|^{1-ml/(n-m)} \leq C(n, m, R).$$

This inequality and (6.22) imply

$$\lim_{j \rightarrow +\infty} \int_0^{|B_R|} \exp(p\beta(n, m)|u_j^*(s)|^{n/(n-m)}) ds = \lim_{j \rightarrow +\infty} \int_{B_R} \exp(p\beta(n, m)|u_j^\sharp|^{n/(n-m)}) dV_g = +\infty. \quad (6.23)$$

We devide our proof into two cases:

Case 1: Suppose that m is even. In this case, we can express $m = 2k$ for some $k \geq 1$. Denote

$$f_j = \Delta_g^k u_j, \quad f = \Delta_g^k u.$$

By passing to a subsequence if necessary, f_j^* converges almost everywhere on $(0, +\infty)$ and converges weakly in $L^{n/m}(0, +\infty)$ to a function g such that $g \geq f^*$. It is evident that

$$\int_0^{+\infty} g(s)^{n/m} ds \leq 1. \quad (6.24)$$

Then Proposition 6.3 and (6.12) give

$$u_j^*(t) \leq \frac{c_{n,k}}{(n\Omega_n^{1/n})^m} \frac{n}{n-m} \int_t^{|B_R|} \frac{f_j^*(s)}{s^{1-m/n}} ds + C(n, m, R)$$

for all $0 < t \leq |B_R|$. Define

$$v_j(t) = \frac{c_{n,k}}{(n\Omega_n^{1/n})^m} \frac{n}{n-m} \int_t^{|B_R|} \frac{f_j^*(s)}{s^{1-m/n}} ds$$

for all $0 < t \leq |B_R|$. Clearly, we have $v_j(|B_R|) = 0$. For any $\delta > 0$, we also have

$$(u_j^*(t))^{n/(n-m)} \leq (1 + \delta)v_j(t)^{n/(n-m)} + C_\delta C(n, m, R)^{n/(n-m)},$$

Choose $\delta > 0$ small enough such that $q = (1 + \delta)p < P_{n,m}(u)$, then we conclude from (6.23) that

$$\lim_{j \rightarrow +\infty} \int_0^{|B_R|} \exp(q\beta(n, m)|v_j(t)|^{n/(n-m)}) ds = +\infty. \quad (6.25)$$

From the definition of v_j , we have

$$v_j(t) \leq \left(\frac{1}{\beta(n, m)} \ln \left(\frac{|B_R|}{t} \right) \right)^{(n-m)/n}. \quad (6.26)$$

We claim that for any $r \in (q, P_{n,m}(u))$, any $j_0 \in \mathbb{N}$, and any $s_0 \in (0, |B_R|)$ there exists $j \geq j_0$ and $s \in (0, s_0)$ such that

$$v_j(s) \geq \left(\frac{1}{r\beta(n, m)} \ln \left(\frac{|B_R|}{s} \right) \right)^{(n-m)/n}. \quad (6.27)$$

Indeed, if this were not true, then there would exist $r \in (q, P_{n,m}(u))$, $j_0 \in \mathbb{N}$ and $s_0 \in (0, |B_R|)$ such that

$$v_j(s) \leq \left(\frac{1}{r\beta(n, m)} \ln \left(\frac{|B_R|}{s} \right) \right)^{(n-m)/n}$$

for all $s \in (0, s_0)$. This implies that

$$\begin{aligned} \int_0^{|B_R|} \exp(q\beta(n, m)|v_j(t)|^{n/(n-m)}) ds &= \int_0^{s_0} \exp(q\beta(n, m)|v_j(t)|^{n/(n-m)}) ds \\ &\quad + \int_{s_0}^{|B_R|} \exp(q\beta(n, m)|v_j(t)|^{n/(n-m)}) ds \\ &\leq \int_0^{s_0} \left(\frac{|B_R|}{s} \right)^{q/r} ds + \int_{s_0}^{|B_R|} \left(\frac{|B_R|}{s} \right)^q ds \end{aligned}$$

$$\leq C(n, m, q, r, s_0, R)$$

for any $j \geq j_0$. This contradicts with (6.25); hence proves our claim (6.27). Thus, up to a subsequence if necessary, we can assume that there exists a sequence $\{s_j\} \subset (0, |B_R|)$ such that $s_j \leq 1/j$ and that

$$v_j(s_j) \geq \left(\frac{1}{r\beta(n, m)} \ln \left(\frac{|B_R|}{s_j} \right) \right)^{(n-m)/n}. \quad (6.28)$$

Given $L > 0$, let us consider the truncation operators T^L and T_L acting on functions v through

$$\begin{cases} T^L(v) = \min\{|v|, L\} \operatorname{sign}(v), \\ T_L(v) = v - T^L(v). \end{cases}$$

It is easy to see that $T^L(f_j^*)$ and $T_L(f_j^*)$ converge almost everywhere to $T^L(g)$ and $T_L(g)$ on $(0, +\infty)$, respectively. Since $\lim_{j \rightarrow +\infty} v_j(s_j) = +\infty$, given any $L > 0$, after passing to a subsequence if necessary, we can assume that $v_j(s_j) > L$ for any j . Then there exists $r_j \in (s_j, |B_R|)$ such that $v_j(r_j) = L$. In the other hand, from the definition of v_j , we have

$$v_j(s_j) \leq \frac{c(n, k+1)}{(n\Omega_n^{1/n})^m} f_j^*(s_j) |B_R|^{m/n},$$

hence $\lim_{j \rightarrow +\infty} f_j^*(s_j) = \infty$. Therefore, by passing again to a subsequence if necessary, we assume that $f_j^*(s_j) > L$ for any j , hence there exists $t_j \in (0, +\infty)$ such that $f_j^*(t_j) = L$ and $f_j^*(s) < L$ for any $s > t_j$. Denote $a_j = \min\{t_j, r_j\}$. We then have

$$\begin{aligned} v_j(s_j) - L &= \frac{c(n, k)}{(n\Omega_n^{1/n})^m} \frac{n}{n-m} \int_{s_j}^{r_j} \frac{f_j^*(s)}{s^{1-m/n}} ds \\ &\leq \frac{c(n, k)}{(n\Omega_n^{1/n})^m} \frac{n}{n-m} \int_{s_j}^{a_j} \frac{f_j^*(s) - L}{s^{1-m/n}} ds + \frac{c(n, k)}{(n\Omega_n^{1/n})^m} \frac{n}{n-m} \int_{s_j}^{r_j} \frac{L}{s^{1-m/n}} ds \\ &\leq \left(\int_{s_j}^{a_j} (f_j^*(s) - L)^{n/m} ds \right)^{m/n} \left(\frac{1}{\beta(n, m)} \ln \frac{a_j}{s_j} \right)^{\frac{n-m}{n}} + \frac{c(n, k+1)}{(n\Omega_n^{1/n})^m} L r_j^{m/n} \\ &\leq \left(\int_0^{+\infty} (T_L(f_j^*))^{n/m} ds \right)^{m/n} \left(\frac{1}{\beta(n, m)} \ln \frac{|B_R|}{s_j} \right)^{\frac{n-m}{n}} + \frac{c(n, k+1)}{(n\Omega_n^{1/n})^m} L |B_R|^{m/n} \end{aligned}$$

The latter estimate and (6.28) implies that for j large enough, we have

$$r^{-(n-m)/n} \leq \left(\int_0^{+\infty} (T_L(f_j^*))^{n/m} ds \right)^{m/n},$$

or equivalently,

$$r^{-(n-m)/m} \leq \int_0^{+\infty} (T_L(f_j^*))^{n/m} ds.$$

Hence, for j large enough

$$1 - r^{-(n-m)/m} \geq \int_0^{+\infty} \left((f_j^*)^{n/m} - (T_L(f_j^*))^{n/m} \right) ds.$$

Letting j to infinity and using the Fatou lemma we have

$$1 - r^{-(n-m)/m} \geq \int_0^{+\infty} \left(g^{n/m} - (T_L(g))^{n/m} \right) ds. \quad (6.29)$$

Next, we shall obtain a contradiction by using (6.29).

Case 1.1: Suppose $\|f\|_{n/m} < 1$. Since

$$\int_0^{+\infty} g^{n/m} ds \geq \int_0^{+\infty} (f^*)^{n/m} ds = \|f\|_{n/m}^{n/m}$$

and

$$\lim_{L \rightarrow +\infty} \int_0^{+\infty} (T_L(g))^{n/m} ds = 0,$$

we can choose a $L > 0$ such that

$$\frac{1 - \|f\|_{n/m}^{n/m}}{1 - \int_0^{+\infty} (g^{n/m} - (T_L(g))^{n/m}) ds} > \left(\frac{r}{P_{n,2k}(u)} \right)^{(n-m)/m}. \quad (6.30)$$

Fix such an L , it follows from (6.29) and (6.30) that

$$\begin{aligned} r &\geq \left(1 - \int_0^{+\infty} (g^{n/m} - (T_L(g))^{n/m}) ds \right)^{-m/(n-m)} \\ &> \frac{r}{P_{n,2k}} (1 - \|f\|_{n/m}^{n/m})^{-m/(n-m)} = r \end{aligned}$$

which is a contradiction.

Case 1.2: Suppose $\|f\|_{n/m} = 1$. Then we must have $\int_0^{+\infty} g^{n/m} ds = 1$. Then we can choose some large $L > 0$ such that

$$\int_0^{+\infty} (g^{n/m} - (T_L(g))^{n/m}) ds > 1 - \frac{1}{2} \left(\frac{1}{r} \right)^{(n-m)/m}.$$

Fix such an L , then we obtain a contradiction since by (6.29) we have

$$1 - r^{-(n-m)/m} \geq \int_0^{+\infty} (g^{n/m} - (T_L(g))^{n/m}) ds > 1 - \frac{1}{2} \left(\frac{1}{r} \right)^{(n-m)/m}.$$

This finishes our proof in case that m is even.

Case 2: Suppose that m is odd. Since the case $m = 1$ was proved in [Kar15]. Using the same argument in [OM14b] (as the one we use in this paper) gives another proof of this case. Hence we will only prove for $m \geq 3$. Therefore, we can write $m = 2k + 1$ for some $k \geq 1$. Denote

$$f_j = \Delta_g^k u_j, \quad f = \Delta_g^k u.$$

Using the Sobolev inequality we have $\|f_j\|_{n/(2k)} \leq C$. Proposition 6.3 and (6.12) gives

$$u_j^*(t_1) - u_j^*(t_2) \leq \frac{c(n, k)}{(n\Omega_n^{1/n})^{2k}} \frac{n}{n-2k} \int_{t_1}^{t_2} \frac{f_j^*(s)}{s^{1-2k/n}} ds + C(n, k).$$

Since $\|\nabla f_j\|_{n/m} \geq \|\nabla f_j^\sharp\|_{n/m}$, we know that

$$\begin{aligned} 1 &\geq \int_{\mathbb{H}^n} |\nabla_g f_j^\sharp|^{n/m} dV_g \\ &= (n\Omega_n)^{n/m} \int_0^{+\infty} |(f^*)'(s)|^{n/m} (\sinh F(s))^{n(n-1)/m} ds. \end{aligned} \quad (6.31)$$

Using integration by parts we obtain

$$\int_{t_1}^{t_2} \frac{f_j^*(s)}{s^{1-2k/n}} ds = \frac{n}{2k} \int_{t_1}^{t_2} (-f_j^*)'(s) s^{2k/n} ds + \frac{n}{2k} t_2^{2k/n} f_j^*(t_2) - \frac{n}{2k} t_1^{2k/n} f_j^*(t_1).$$

From (6.31) and the fact that $\sinh F(s) \geq (s/\Omega_n)^{1/n}$ we easily deduce that $t^{2k/n} f_j^*(t) \leq C$ for some C independent of j . Consequently, we obtain

$$u_j^*(t_1) - u_j^*(t_2) \leq \frac{c(n, k+1)}{(n\Omega_n^{1/n})^{2k}} \int_{t_1}^{t_2} (-f_j^*)'(s) s^{2k/n} ds + C(n, k), \quad (6.32)$$

for some $C(n, k)$ depending only on n and k . Note that (6.32) plays the role of (6.16) in our proof below when m is odd. Our proof proceeds along the same line as in the case when m is even; hence we limit ourselves to sketch the proof. Define

$$v_j(t) = \frac{c(n, k+1)}{(n\Omega_n^{1/n})^{2k}} \int_t^{|B_R|} (-f_j^*)'(s) s^{2k/n} ds$$

for $t \in (0, |B_R|)$. Then for $q \in (p, P_{n,m}(u))$ we then have

$$\lim_{j \rightarrow +\infty} \int_0^{|B_R|} \exp(q\beta(n, m)|v_j(t)|^{n/(n-m)}) ds = +\infty. \quad (6.33)$$

For any $r \in (q, P_{n,m}(u))$, for any $j_0 \in \mathbb{N}$ and any $s_0 \in (0, |B_R|)$ there exist $j \geq j_0$ and $s \in (0, s_0)$ such that

$$v_j(s) \geq \left(\frac{1}{r\beta(n, 2k)} \ln \left(\frac{|B_R|}{s} \right) \right)^{(n-2k)/n}. \quad (6.34)$$

Indeed, if this does not hold true we will obtain a contradiction as in the case m even since

$$v_j(t) \leq \left(\frac{1}{\beta(n, 2k+1)} \ln \left(\frac{|B_R|}{t} \right) \right)^{(n-2k-1)/n}$$

for all $t \in (0, |B_R|)$. Thus, up to a subsequence, there exists a sequence $\{s_j\} \subset (0, |B_R|)$ such that $s_j \leq 1/j$ and that

$$v_j(s_j) \geq \left(\frac{1}{r\beta(n, 2k+1)} \ln \left(\frac{|B_R|}{s_j} \right) \right)^{(n-2k-1)/n}. \quad (6.35)$$

Since

$$v_j(s_j) \leq \frac{c(n, k+1)}{(n\Omega_n^{1/n})^{2k}} |B_R|^{2k/n} (f_j^*(s_j) - f_j^*(|B_R|)),$$

we conclude that

$$\lim_{j \rightarrow +\infty} v_j(s_j) = \lim_{j \rightarrow +\infty} f_j^*(s_j) = +\infty.$$

Therefore, given $L > 0$, by passing to a subsequence, we can assume that $v_j(s_j) > L$ and $f_j^*(s_j) > L$. Hence there exists $r_j \in (s_j, |B_R|)$ and $t_j \in (s_j, +\infty)$ such that $v_j(r_j) = L$ and $f_j^*(t_j) = L$. Denote $a_j = \min\{t_j, r_j\}$, we have

$$\begin{aligned} v_j(s_j) - L &= \frac{c(n, k+1)}{(n\Omega_n^{1/n})^{m-1}} \int_{s_j}^{r_j} (-f_j^*)'(s) s^{(m-1)/n} ds \\ &= \frac{c(n, k+1)}{(n\Omega_n^{1/n})^{2k}} \int_{s_j}^{a_j} (-f_j^*)'(s) s^{(m-1)/n} ds + \frac{c(n, k+1)}{(n\Omega_n^{1/n})^{m-1}} \int_{a_j}^{r_j} (-f_j^*)'(s) s^{(m-1)/n} ds \\ &\leq \frac{c(n, k+1)}{(n\Omega_n^{1/n})^m} \left((n\Omega_n)^{n/m} \int_{s_j}^{a_j} |(f_j^*)'(s)|^{n/m} (\sinh F(s))^{n(n-1)/m} ds \right)^{m/n} \\ &\quad \times \left(\int_{r_j}^{a_j} (\sinh F(s))^{-n(n-1)/(n-m)} s^{(m-1)/(n-m)} ds \right)^{(n-m)/n} \\ &\quad + \frac{c(n, k+1)}{(n\Omega_n^{1/n})^{m-1}} a_j^{(m-1)/n} (f_j^*(a_j) - f_j^*(r_j)) \\ &\leq \|T_L(f_j^\sharp)\|_{n/m} \left(\frac{1}{\beta(n, m)} \ln \left(\frac{|B_R|}{s_j} \right) \right)^{(n-m)/n} + \frac{c(n, k+1)}{(n\Omega_n^{1/n})^{m-1}} |B_R|^{(m-1)/n} L, \end{aligned}$$

here we use the estimate $\sinh F(s) \geq (s/\Omega_n)^{1/n}$ and the facts that if $t_j < r_j$ then $f_j^*(a_j) - f_j^*(r_j) \leq L$ while if $t_j \geq r_j$ then $f_j^*(a_j) - f_j^*(r_j) = 0$. Hence, for j large enough we then have from (6.35) that

$$r^{-(n-m)/m} \leq \int_{\mathbb{H}^n} |\nabla_g T_L(f_j^\sharp)|^{n/m} dV_g.$$

Note that $T_L(f_j^\sharp) = (T_L(f_j))^\sharp$; thus

$$r^{-(n-2k-1)/(2k+1)} \leq \int_{\mathbb{H}^n} |\nabla_g T_L(f_j^\sharp)|^{n/(2k+1)} dV_g \leq \int_{\mathbb{H}^n} |\nabla_g T_L(f_j)|^{n/(2k+1)} dV_g.$$

Notice that

$$\int_{\mathbb{H}^n} |\nabla_g T_L(f_j)|^{n/m} dV_g + \int_{\mathbb{H}^n} |\nabla_g T^L(f_j)|^{n/m} dV_g = \int_{\mathbb{H}^n} |\nabla_g f_j|^{n/m} dV_g.$$

Then, we have for j large enough

$$1 - r^{-(n-m)/m} \geq \int_{\mathbb{H}^n} |\nabla_g T^L(f_j)|^{n/m} dV_g.$$

We have $T^L(f_j)$ converges almost everywhere to $T^L(f)$ on \mathbb{H}^n . Moreover, $\{T^L(f_j)\}_j$ is bounded sequence in $W^{1,n/m}(\mathbb{H}^n)$, by passing to a subsequence if necessary, we assume that

- $T^L(f_j)$ converges weakly to a function g on $W^{1,n/m}(\mathbb{H}^n)$ and
- $T^L(f_j)$ converges to g in $L_{\text{loc}}^p(\mathbb{H}^n)$ for any $p < n/(2k)$ by the Rellich-Kondrachov theorem.

This shows that $g = T^L(f)$, hence by the weak lower semi-continuity of the $L^{n/m}$ -norm of gradient, we have

$$1 - r^{-(n-m)/m} \geq \int_{\mathbb{H}^n} |\nabla_g T^L(f)|^{n/m} dV_g, \quad (6.36)$$

which is similar to (6.29).

Case 2.1: Suppose $\|\nabla_g f\|_{n/m} < 1$. Since

$$\lim_{L \rightarrow +\infty} \int_{\mathbb{H}^n} |\nabla_g T^L(f)|^{n/m} dV_g = \int_{\mathbb{H}^n} |\nabla_g f|^{n/m} dV_g,$$

we can choose some large $L > 0$ such that

$$\frac{1 - \|\nabla_g f\|_{n/m}^{n/m}}{1 - \|\nabla_g T^L(f)\|_{n/m}^{n/m}} > \left(\frac{r}{P_{n,2k+1}(u)} \right)^{(n-m)/m}. \quad (6.37)$$

Fix such an $L > 0$. Using (6.36) for this L we have

$$\begin{aligned} r &\geq (1 - \|\nabla_g T^L(f)\|_{n/m}^{n/m})^{-m/(n-m)} \\ &> \frac{r}{P_{n,m}(u)} (1 - \|\nabla_g f\|_{n/m}^{n/m})^{-m/(n-m)} = r, \end{aligned}$$

which is a contradiction.

Case 2.2: Suppose $\|\nabla_g f\|_{n/m} = 1$. Then we can choose $L > 0$ such that

$$\|\nabla_g T^L(f)\|_{n/m}^{n/m} > 1 - \frac{1}{2} \left(\frac{1}{r} \right)^{(n-m)/m}.$$

Fix such an $L > 0$. Using (6.36) for this L we obtain a contradiction since

$$1 - r^{-(n-m)/m} \geq \|\nabla_g T^L(f)\|_{n/m}^{n/m} > 1 - \frac{1}{2} \left(\frac{1}{r} \right)^{(n-m)/m}.$$

This finishes our proof when m is odd.

6.4. **The sharpness of (AMT_{CC}^H).** It remains to check the sharpness of the exponent $P_{n,m}(u)$ in Theorem 1.6. We will show that for any $\alpha \in (0, 1)$, there exists a sequence $\{u_j\}_j \subset W^{m,n/m}(\mathbb{H}^n)$ and $u \in W^{m,n/m}(\mathbb{H}^n)$ such that

- $\|\nabla_g^m u_j\|_{n/m} = 1$, $\|\nabla_g^m u\|_{n/m} = \alpha$,
- $u_j \rightharpoonup u$ in $W_0^{m,n/m}(\mathbb{H}^n)$, and
- $u_j \rightarrow u$ almost everywhere on \mathbb{H}^n

such that

$$\lim_{j \rightarrow +\infty} \int_{\mathbb{H}^n} \Phi_{n,m}(\beta(n, m)(1 - \alpha^{n/m})^{-m/(n-m)} |u_j|^{n/(n-m)}) dx = +\infty.$$

For $j \geq 2$, we define

$$v_j(x) = \begin{cases} \left(\frac{\ln j}{\beta(n, m)} \right)^{1-m/n} + \frac{n\beta(n, m)^{m/n-1}}{2(\ln j)^{m/n}} \sum_{l=1}^{m-1} \frac{(1 - j^{2/n}|x|^2)^l}{l} & \text{if } 0 \leq |x| \leq j^{-1/n}, \\ -n\beta(n, m)^{m/n-1}(\ln j)^{-m/n} \ln |x| & \text{if } j^{-1/n} \leq |x| < 1, \\ \xi_j(x) & \text{if } 1 \leq |x| \leq 2, \end{cases}$$

where $\xi_j \in C_0^\infty(B_2)$ are radial functions which are chosen such that $\xi_j = 0$ on ∂B_1 and ∂B_2 , and for $l = 1, 2, \dots, k-1$

$$\frac{\partial^l \xi_j}{\partial r^l} \Big|_{\partial B_1} = (-1)^l (l-1)! \beta(n, m)^{m/n-1} (\ln j)^{-m/n}, \quad \frac{\partial^l \xi_j}{\partial r^l} \Big|_{\partial B_2} = 0,$$

and ξ_j , $|\nabla^l \xi_j|$ and $|\nabla^k \xi_j|$ are all $O((\ln j)^{-m/n})$. The choice of these functions is inspired from [Zha13, Section 3].

Consider the function $w_j(x) = v_j(3x)$ where $x \in \mathbb{H}^n$. Clearly, $w_j \in W^{m,n/m}(\mathbb{H}^n)$ with support in $\overline{B}_{2/3}$. An easy computation shows that

$$1 \leq \int_{\mathbb{H}^n} |\nabla^m v_j(x)|^{n/m} dx \leq 1 + O((\ln j)^{-1});$$

hence

$$1 - \frac{c}{\ln j} \leq \|\nabla_g^m w_j\|_{n/m}^{n/m} \leq 1 + \frac{C}{\ln j}, \quad (6.38)$$

for some constant $C, c > 0$ independent of j . Setting

$$\tilde{w}_j = w_j / \|\nabla_g^m w_j\|_{n/m},$$

we have:

- $\tilde{w}_j \rightharpoonup 0$ weakly on $W^{m,n/m}(\mathbb{H}^n)$ and
- $\tilde{w}_j \rightarrow 0$ almost everywhere on \mathbb{H}^n .

Taking a function $v \in C_0^\infty(B_1)$ such v is constant in $B_{2/3}$ and $\|\nabla_g^m v\|_{n/m} = \alpha$. Then we define

$$u_j = v + (1 - \alpha^{n/m})^{m/n} \tilde{w}_j.$$

Clearly $u_j \in W_0^{m,n/m}(\Omega)$ and $\|\nabla_g^m u_j\|_{n/m} = 1$ for all $j \geq 2$ since the supports of $\nabla_g^m v$ and $\nabla_g^m \tilde{w}_j$ are disjoint and $u_j \rightharpoonup v$ in $W^{m,n/m}(\mathbb{H}^n)$. Replacing v by $-v$ if necessary, we can assume that $v \geq A$ on $B_{2/3}$ for some $A > 0$. Then we can estimate

$$\begin{aligned} & \int_{\mathbb{H}^n} \Phi_{n,m}(\beta(n, m)(1 - \alpha^{n/m})^{-m/(n-m)} |u_j|^{n/(n-m)}) dV_g \\ & \geq \int_{|x| \leq j^{-1/n}} \Phi_{n,m} \left(\frac{\beta(n, m)}{(1 - \alpha^{n/m})^{m/(n-m)}} \times \right. \\ & \quad \left. \left(A + \frac{(1 - \alpha^{n/m})^{m/n}}{(1 + C/\ln j)^{m/n}} \left(\frac{\ln j}{\beta(n, m)} \right)^{1-m/n} \right)^{n/(n-m)} \right) dV_g \end{aligned}$$

$$\geq C' \omega_n \exp \left(\left(C + \frac{(\ln j)^{(n-m)/n}}{(1 + C/\ln j)^{m/n}} \right)^{n/(n-m)} - \ln j \right),$$

for some constant $C, C' > 0$ which are independent of j . It is easy to see that there exists a constant $0 < C_1 < C$ and j_0 such that for any $j \geq j_0$, we have

$$\exp \left(\left(C + \frac{(\ln j)^{(n-m)/n}}{(1 + C/\ln j)^{m/n}} \right)^{n/(n-m)} - \ln j \right) \geq \exp \left(\left(C_1 + (\ln j)^{(n-m)/n} \right)^{n/(n-m)} - \ln j \right),$$

hence

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \int_{\mathbb{H}^n} \exp(\beta(n, m)(1 - \alpha^{n/m})^{-m/(n-m)} |u_j|^{n/(n-m)}) dV_g \\ \geq \lim_{j \rightarrow +\infty} \exp \left(\left(C_1 + (\ln j)^{(n-m)/n} \right)^{n/(n-m)} - \ln j \right) = +\infty. \end{aligned}$$

This proves the sharpness of $(\text{AMT}_{CC}^{\text{H}})$ as claimed.

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